THE ROLE OF CO-CONSTRUCTION OF THE CLOCK MODEL IN THE DEVELOPMENT OF FRACTIONAL UNDERSTANDING IN A GRADE 4/5 CLASSROOM

by

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Abstract

The focus of this case study was to determine the impact of the co-construction and use of the clock model in the development of fractional understanding in a Grade 4/5 classroom. A pretest, instruction, midtest, intervention, posttest, retention test sequence was used. The teaching unit was developed to address six areas of student difficulty with fractions identified in the literature: fractional parts are equal size portions, fraction symbols, improper fractions, size of unit fractions, estimation and comparison, and equivalent fractions. The clock model was co-constructed by the students with the teacher in the second half of the unit. Students used the clock model in four different forms: as an area model, as whole numbers of minutes, as equivalent fractions, and as an open clock. Although some students did not adopt the clock model, others used it with great success. Most students moved beyond using only landmark fractions of an hour to being able to work with thirds and sixths of an hour. The clock model helped some students to better understand the relationships between equivalent fractions. Recommendations are discussed.

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CHAPTER ONE

Introduction

Context

Mathematics education is changing. There are a variety of factors spurring on this transformation, including knowledge gained from research, the prevalence of technology, and the influence of the National Council of Teachers of Mathematics (NCTM)[1] (Battista, 1999; Van de Walle, Folk, Karp & Bay-Williams, 2011).

In the United Sates the NCTM published its *Curriculum and Evaluation Standards for School Mathematics* in 1989, closely followed by the publication of a companion document, *Professional Standards for Teaching Mathematics* in 1991. Implementation of these two influential documents marked the beginning of the "reform era" in mathematics education. The NCTM-influenced new curriculum documents published by the Ontario Ministry of Education followed, first in 1997, and then replaced by *The Ontario Curriculum, Grades 1-8: Mathematics* in 2005. These documents advocated shifts in the classroom environment, reflecting the belief that teaching needed to change in order to improve student learning. What was it about the teaching of mathematics that needed to change?

Numerous studies have shown *traditional* methods of teaching mathematics to be ineffective, and even harmful, because they discourage mathematical reasoning (Battista, 1999; Kamii & Dominick, 1998). The traditional view sees mathematics as a set of computational skills and procedures to be learned through memorization. In a traditional mathematics class the teacher delivers the lesson, the students copy and practice. In contrast, Battista (1999) summarized *reform* teaching methods as occurring in classrooms where:

[1] NCTM is a U.S.-based organization composed of mathematics educators from the United States and Canada.

> teachers provide students with numerous opportunities to solve complex and interesting problems; to read, write, and discuss mathematics; and to formulate and test the validity of personally constructed mathematical ideas so that they can draw their own conclusions. Students use demonstrations, drawings, and real-world objects—as well as formal mathematical and logical arguments—to convince themselves and their peers of the validity of their solutions. (pp. 427-428)

In addition to realizing that there were difficulties caused by traditional teaching methods and that classroom practice should be altered, researchers were drawing on the theory of *constructivism* as the theoretical foundation of this shift.

Constructivism

Constructivism is a theory about knowledge and learning, different from other theories of learning such as behaviourism. In constructivism, knowledge is viewed as being personally constructed by learners as they try to make sense of situations, not as existing independently and absolutely as "truths to be transmitted or discovered" (Fosnot, 2005, p. ix). Students learn—construct new knowledge based on prior knowledge—by reflecting on, or actively thinking about, an idea, rather than passively absorbing it unaltered from the teacher. Because constructivism is about learning, it is a theory with important implications for the classroom. Fosnot and Perry (2005) contend that "teachers need to allow learners to raise their own questions, generate their own hypotheses and models as possibilities, test them out for viability, and defend and discuss them in communities of discourse and practice" (p. 34).

Refinements to Reform Instruction

Researchers continue to refine the conceptualization of effective reform instruction initially outlined in the NCTM Standards. As one of a number of refinements, some researchers shifted their focus from the importance of using mathematical manipulatives as a means to deepening student understanding to the more nuanced contention that models must be developed in a familiar context through the activity of modeling (Gravemeijer, 2002). Researchers came to believe that if manipulatives were used as models, by essentially being given to the students by the teacher rather than being co-constructed by the students together with their teachers, then students often misunderstood the mathematics; that is, they did not 'see' mathematical truths in the manipulative in the way the teacher did (Ball, 1992). This has implications for those areas of mathematics where manipulatives as models have played a central role, and is the case, for example, with the teaching of fractions. It may be that the choice and method of introduction of mathematical models can play an important role in the improvement of understanding of fractions, a particularly poorly understood area of mathematics.

Purpose of the Study

The purpose of this study was to investigate the co-construction and use of the clock model by students in the context of a teaching unit on fractions. I teach mathematics primarily to adult learners who often struggle with learning mathematics and I am in the process of reforming my teaching practice. I am neither an elementary classroom teacher nor a district consultant. I wanted to be able to observe the teaching of the fractions unit in a strong reform classroom environment. My thesis advisor put me in touch with the classroom teacher, someone who had completed her own master's degree in education and therefore had an understanding of research. The fractions unit was taught using a reform method approach, the customary mathematics teaching approach used by the teacher. This study was designed to examine the process and impact on student understanding of a teacher and her students co-constructing a model for comparing and adding fractions. A pretest was administered as a diagnostic tool to ascertain the students' understanding and use of tools and representations prior to instruction. The unit on fractions was taught with an emphasis on problem solving and student-generated solutions. The midtest was administered part way through the unit, before the clock model was co-constructed. The results of the posttest were compared with the midtest and pretest to determine the impact resulting from the co-construction and use of the clock model.

Research Question

What is the impact of the co-construction and use of the clock model on students' understanding of fractions?

Significance of the Study

Although there is a substantial body of research pertaining to the topic of learning fractions within reform mathematics instruction, there is nonetheless much more to be accomplished (Lamon, 2007), including the topic of effective modelling. As Gravemeijer (2002) contended, there is "a growing interest within the mathematics community in the role of symbolizing and modeling" (p. 7). Despite this interest there is a lack of classroom-based research on these topics. Indeed, searches of ERIC, CBCA, and Professional Development Collection education databases using the key words "fractions" and "clock" yielded only a few teaching suggestions for connecting fractions to telling time on the analog clock (Friederwitzer & Berman, 1999) and relating the hours of the day to fractions (May, 2000). Finally, an article by Chick, Tierney and Storeygard (2007) contains a description of two students' understandings as they solved traditional fraction problems using a clock face. No published research on the use of the clock model in the manner suggested by Fosnot and Dolk (2002) was located. Thus this study addresses a current gap in the literature.

This research was a case study of a class of Grade 4/5 students as they learned about fractions. It provides an in-depth exploration of their learning. This study contributes information that will help guide mathematics teachers and educators and will shed light on, as Gravemeijer, Lehrer, van Oers, and Verschaeffel (2002) wondered, “how to support and guide this process of knowledge construction without interfering with students’ initiative and intellectual autonomy” (p. i).

Contribution to the Community

Teachers are interested in improving their practice. The classroom teacher and I will collaborate to prepare a workshop for teachers now that the study has been completed. Furthermore, the use of the lesson videotapes gathered as data in this study will contribute to in-service teacher professional development and may lead to increased student success as a result.

Limitations of the Study

Some aspects of the study should be considered; namely, the design of the research project and the testing instruments. As a case study, this research provides a rich description of the learning that occurred in one classroom. Most of the time, case studies are undertaken to understand the specifics of a particular case, not to generalize to other cases (Stake, 1995). In this instance it is clear that one classroom will not be representative of all Grade 4 and Grade 5 students. This study was not designed to make a comparison between two different teaching methods or models in order to make a judgment about which might be superior. The intent was to look at the impact of the co-construction of the clock model and describe how it supports or does not support student learning.

Pretest questions were selected to give baseline information about students’ understanding of fractions before instruction, including known areas of difficulty. The number of questions on the subsequent tests was reduced out of concern that the pretest was too long for the students. Different questions were selected for the subsequent tests in order to gain specific information about the use of the clock model. As a result, not all areas of difficulty were represented in the items on subsequent tests.

CHAPTER TWO

Literature Review

Introduction

Learning about fractions is a challenge for students (Lamon, 2007; Van de Walle et al., 2011). Most of the time, fractions are taught in the same way mathematics has been traditionally taught. The emphasis is on learning algorithms, which are set procedures that are followed in order to complete a calculation. For example, an analysis by Cramer, Post and delMas (2002) of a commercial fourth- and fifth-grade textbook used in one school district revealed the emphasis was on gaining procedural competency, and distressingly, that "symbol manipulation seemed to be an end in itself, independent of context and physical models" (p. 139). Researchers have been concerned that under traditional instruction students can become proficient in carrying out rote procedures but, because the instruction focuses on computation rather than on understanding, students may later lose this apparent mastery (e.g., Aksu, 1997). Researchers believe that when procedural knowledge is taught before or without developing conceptual knowledge, children may not see the connection between the two and the result is that they must memorize the algorithms. Conversely, researchers have speculated that when children can instead build on their intuitive knowledge of fractions gained by personal experience and eventually come to rely less on the specific context, they continue to make sense of fractions. Once children have developed conceptual knowledge of fractions, they can connect concepts to procedures and make use of formal symbols and algorithms (Sharp, Garofalo & Adams, 2002).

Fractions have multiple meanings and interpretations. Researchers and mathematics educators generally agree that there are five main notions of fractions: fractions as parts of wholes or sets; fractions as quotients; fractions as ratios; fractions as operators; and fractions as measures (Behr, Harel, Post & Lesh, 1992; Lamon, 2007). This study and the literature review focus mainly on part-whole fractional relationships.

A general recommendation for improving the teaching of fractions is to help students develop conceptual understanding rather than concentrating on symbols and procedures. What are the specific areas that have typically caused problems for students as well as the recommendations for remediation found in the literature?

Instructional Difficulties

Researchers have documented many problems that students have with learning fractions and have tested teaching strategies that could help students to avoid, or at least minimize, the difficulties. Six important concepts, the problems faced, and some suggested instructional solutions are examined below.

Fractional parts are equal-size portions

A key mathematical idea underpinning the development of fraction concepts is the recognition that “fractional parts are equal shares or equal-sized portions of a whole or unit” (Van de Walle, 2007, p. 293). Researchers have found that children do not always realize that the whole must be partitioned into pieces of equal size (e.g., Pothier & Sawada, 1983). In their study, Reys, Kim and Bay (1999) posed a series of open-ended questions to a class of fifth graders who had recently completed a six-week unit on fractions. When asked to describe $^{2}/_{5}$, seven out of the 20 students gave responses that revealed they held the misconception that the whole did not need to be partitioned into pieces of equal size. Three children made inaccurate circle diagrams to represent the fraction $^{2}/_{5}$ and were not concerned that the pieces were not of equal size, even when questioned about the sections as if they were pieces of pizza. About one third of the class held a misconception about this fundamental fractions concept; clearly something was lacking in the instruction they had received.

Although many would suggest beginning with students' intuitive knowledge, Pothier and Sawada (1983) found in the context of cutting and sharing pieces of birthday cake, that some children were more concerned with making sure the number of pieces for each person was the same, rather than fairness of the size of the pieces. Therefore, surprisingly, they observed that informal knowledge is not always a reliable starting point. Most other researchers and math educators, however, have found otherwise. Many advocate the use of a fair-sharing context (e.g., Empson, 2002; Flores & Klein, 2005; Fosnot & Dolk, 2002; Sharp et al., 2002). With fair-sharing tasks, it is best to make use of items which can easily be subdivided, such as brownies, pizzas, sandwiches, or candy bars (Van de Walle, 2007), in a context which is personal for the children (Fosnot & Dolk, 2002; Sharp et al., 2002).

Additionally, in order to help students construct the importance of equal sized portions in fractions, Reys, Kim, and Bay (1999) suggested that teachers need to provide more opportunities for students to work with visual representations of fractions, such as shading fractional parts of shapes, locating fractions on a number line, or separating tiles, and recommend that teachers revisit these representations at the fifth-grade level in order for students to fully conceptualize fractions before they move on to procedures with fractions. Researchers with the Rational Number Project (Cramer et al., 2002) showed that the use of, and translations between, multiple physical manipulatives, symbols, pictures and contexts during initial fraction learning is effective in helping children to construct accurate mental images of fractions.

Understanding fraction symbols

Researchers have found that the symbolism used to represent fractions is a complex convention and can be misleading (Van de Walle, 2007). For example, in Mack's (1995) research with third and fourth graders, students "thought the numerator represented the number of wholes, or units, being considered and the denominator represented the number of parts in each whole" (p. 430). Mack tried to help students relate fraction symbols to fractions used in the context of real-world problems they had solved verbally, but it was very difficult. Students drew on previous knowledge of whole numbers rather than on their informal knowledge of fractions when writing and explaining fraction symbols. After Mack (1995) showed the children how to write mixed numbers and explained their meaning in the context of real-world situations, the children were able to write mixed numerals and stopped saying that the numerator was the number of wholes under consideration.

Since fraction symbolism is a convention, Van de Walle (2007) advised delaying its introduction until students had worked with fractional ideas and then simply telling students the way fractions are written, but to make the convention very clear by leading the following demonstration. He recommended displaying several collections of fractional parts to the class, including some improper fractions, having the students count the parts together, writing the correct fractional symbol for each one, and then posing two questions to the class: "What does the bottom number in a fraction tell us? What does the top number in a fraction tell us?" (p. 299). For part-whole relationships, the important point is that children grasp that the top number is the counting number and the bottom number tells what is being counted. Use of the terms numerator and denominator will not help children understand the meaning of the symbolism.

Improper fractions

Another misconception held by many students is that fractions are always less than one whole. In Mack's (1990) study of sixth graders, she found that students were not able to identify the unit when presented with fractions greater than one whole concretely or pictorially because of their belief that fractions are always less than one whole. However, when the situation was given a real-world context, such as pizzas, students were able to interpret the unit correctly. Tzur (1999) was puzzled by the inability of fourth grade students to think about fractions greater than one. Working together with sticks of various lengths in a computer microworld, the students would change the unit after producing nonunit fractions by iterating a unit fraction. (A unit fraction is a single fractional part and has a numerator of one; a nonunit fraction has a numerator greater than one). Tzur related this particular observation from his teaching experiment:

> For example, they regarded the nonunit fraction produced by the iteration of $^{1}/_{5}$ four times as $^{4}/_{5}$ or the unit produced by iteration of $^{1}/_{5}$ five times as $^{5}/_{5}$, but when they iterated the same $^{1}/_{5}$ six times, they thought about the resultant stick as $^{6}/_{6}$ and each part was then regarded as $^{1}/_{6}$. (p. 397)

It is likely that this misconception has its roots in the way fractions are commonly introduced in North American textbooks, with geometric figures displaying shaded parts of the whole, as well as in the language of fractions. Flores and Klein (2005) reminded us that the school language of fractions differs from everyday usage. In daily use, the term "fraction" means a part less than one whole. In mathematics, the term fraction refers to the quotient of two quantities. In addition, the term "improper fraction" may confuse students by suggesting that there is something unacceptable about this type of fraction. It is important to give children the opportunity to count pieces, so that $^{3}/_{4}$ can be viewed "not only as three out of four pieces but also as three pieces of size $^{1}/_{4}$ each" (p. 456). This can be extended, for example, to counting seven pieces of size $^{1}/_{4}$ each. Also, the posing of fair-sharing problems such as sharing seven brownies among four people will elicit answers that include $^{7}/_{4}$, among others. Solving problems like this will assist students in constructing improper fractions more easily. Van de Walle et al. (2011) recommended saying "fractions greater than one" or just simply "fractions" rather than using the term improper fractions (p. 305). And, as mentioned above, improper fractions should be included when introducing fraction symbols (Van de Walle et al., 2011). Indeed Kamii and Clark (1995) reasoned that proper and improper fractions, as well as mixed numbers, should all "be involved from the beginning so that children will think about parts and wholes at the same time" (p. 375).

Size of unit fractions

Children often have difficulty conceptualizing the relative size of unit fractions (Van de Walle, 2007). Mack (1990) found, for example, when sixth grade students were asked which fraction is bigger, $^{1}/_{6}$ or $^{1}/_{8}$, most of them chose $^{1}/_{8}$. Their explanations for choosing $^{1}/_{8}$ as the larger fraction suggested that they were trying to apply rules for whole numbers to fractions. When the same comparison was placed in the context of slices of pizza however, the students drew on their informal knowledge and answered with $^{1}/_{6}$. "The inverse relationship between number of parts and size of parts cannot be told but must be a creation of each student's own thought process" (Van de Walle, 2007, p. 304). Alluding to constructivism, Van de Walle (2007) made the case that understanding cannot simply be 'given'; rather, it must be constructed by the learner. In order to overcome the misconception, he suggested teachers have children go through an activity of putting a list of unit fractions in order from least to greatest, defending the way they ordered the fractions, and explaining their ideas by using models (p. 304).

Alternatively, Fosnot and Dolk (2002) re-asserted that the misconception can usually be avoided “when fractions are introduced right from the start within fair-sharing contexts, as division” (p. 56). They explained that when fractions are introduced as shaded parts of whole entities, as they traditionally are, this is introducing fractions as a number within a measurement model, which is quotative division. When the denominator is larger, there are more pieces. In this context “learners often assume that the greater the denominator the greater the amount” (Fosnot & Dolk, 2002, p. 56). In a fair-sharing context however, children understand (by physically sharing wholes) that the greater the number of pieces, the smaller the amount, and avoid the problem of assuming that a greater denominator signifies a greater amount.

Estimation and comparison: The need for fraction number sense

Another difficulty that arises from traditional fraction instruction is that many students are not able to estimate a sum or correctly compare the sizes of fractions (Cramer & Henry, 2002; Cramer et al., 2002; Reys et al., 1999). This situation signifies a lack of fraction number sense. Students need to develop an awareness of the approximate size of a fraction in order to estimate. Traditional textbook-based instruction does not include estimation; students are taught to rely on rote procedures such as rewriting fractions with common denominators or using cross-multiplication to be able to compare fractions.

One way to overcome the inability to estimate is to teach about important reference points, or benchmarks (Huinker, 1998). Essential fraction benchmarks are 0, $^1/_2$, and 1 (Van de Walle, 2007). Fosnot and Dolk (2002) also recognized the importance of using benchmark or landmark fractions as a strategy. Teachers need to intentionally model the use of fraction benchmarks for their students and show that it is a useful process that they will use outside of math class (Reys et al., 1999).

A major study carried out by researchers with the Rational Number Project (RNP) demonstrated that there is a method of teaching fractions using manipulatives which helps children build fraction number sense. In 2002, Cramer et al. reported on their research which measured and contrasted the effects of two different curricula on fractions achievement and thinking of over 1600 fourth and fifth grade students. One group used district-approved commercial curriculum (CC); the other used the RNP curriculum. The RNP curriculum emphasized "the extended use of multiple physical models and translations within and between other modes of representation—pictures, written symbols, verbal symbols, and real-world contexts" (pp. 137-138). In the commercial curriculum the emphasis was on gaining procedural competency. Written tests and task-based interviews were used to measure achievement and reveal students' thinking.

On the written tests, RNP students had significantly higher scores on four of the six subscales: concepts, order, transfer, and estimation (Cramer et al., 2002). Unexpectedly, no significant differences were found between the groups of students on equivalence items or symbolic operations tasks. The researchers had expected CC students to outperform RNP students on operations tasks requiring exact answers, since CC students had spent significantly more time on this topic. Thus, as the authors state, "RNP students' development of procedural knowledge was apparently not impeded despite their having devoted very limited classroom time to it" (p. 138).

The interview data revealed differences between the two groups in students' thinking (Cramer et al., 2002). The RNP students displayed higher percentages of conceptually oriented answers than the CC students. They used mental models of fractions, chiefly the circle model, to determine relative sizes of fractions for the purpose of putting them in order, and also extended the use of the mental representations to estimating sums and differences. In contrast, the CC students relied on procedures such as finding least common denominators or cross products to obtain solutions. These results clearly reflected the curriculum used by each group. Consequently, it can be concluded that working with physical models assists children in developing fraction number sense; students created mental images of fractions which in turn helped them to understand fraction size.

Equivalent fractions

Another crucial yet problematic area for students is equivalent fractions (Kamii & Clark, 1995). This is the first time in their mathematics learning students encounter a situation where "a fixed quantity can have multiple names (actually an infinite number)" (Van de Walle et al., 2011, p. 310). For Huinker (1998), understanding that "a specific amount can have many names" (p. 172) is a critical aspect of fraction knowledge. Fosnot and Dolk (2002) theorized that two *big ideas*[2] must be constructed by children in order to understand equivalent fractions: "for equivalence the ratio must be kept constant" and "pieces don't have to be congruent to be equivalent" (pp. 136-137). In the first of these two big ideas, understanding ratio means understanding the relationship within a fraction, between the numerator and the denominator, and then extending this relationship across two fractions. In the second big idea, congruent pieces would be pieces that are the same shape and size. For area models, two fractions can be equivalent, even if the pieces are not the same shape, as long as they are the same size. For example, in sharing submarine sandwiches, three pieces of $^{1}/_{4}$ of a sub are the same amount as $^{1}/_{2}$ plus $^{1}/_{4}$ of a sub: "the quantity stays the same even though the pieces look different" (p. 56). Fosnot and Dolk's contention is substantiated in the research. An exploratory study found that

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[2] Big ideas are "the central, organizing ideas of mathematics – principles that define mathematical order" (Schifter & Fosnot, 1993, p. 35).

students were able to explain equivalent fractions when presented to them as visual geometric area models, especially circles, but had difficulties with numerical symbolic notation (Jigyel & Afamasaga-Fuata'i, 2007). One conclusion reached by the researchers is that children need to develop their understanding of the idea of a fraction as the relationship between the numerator and the denominator.

The general approach to equivalent fractions encouraged by Van de Walle et al. (2011) is to have students use a variety of models (area, length, and set) to generate different names for the same fractions. Lamon (1996) suggested that partitioning activities should continue to be given throughout middle school so that students develop increasingly sophisticated partitioning strategies, rather than thinking of partitioning as a simple introductory level activity for students in the third grade only. When comparing the value of partitioning by physically cutting pieces with the value of making drawings, Lamon stated: "there is a greater chance that notions of equivalence will be visually induced in the paper-and-pencil tasks" (p. 189). Even so, Kamii and Clark (1995) showed that children's figurative knowledge (based on what is observable) can be in conflict with the operative knowledge (based on relationships, which are not observable). This conflict may cause children to say that a triangular half of a square is larger than a rectangular half of a square of the same size. Students have trouble constructing the idea that one half is equal to one half if the regions are shaped differently. Lamon (2002) asserted that students are able to generate equivalent fractions using the reasoning of *unitizing*. Unitizing is the "process of mentally constructing different-sized chunks in terms of which to think about a given commodity" (p. 80). Lamon wrote that one advantage of unitizing is that children can reason in this way about fractions "even before they have the physical coordination to be able to draw fractional parts accurately" (p. 82). There does not seem to be a consensus in the literature on the best way to teach equivalent fractions.

Summary of instructional difficulties and remedies

Throughout the discussion of difficulties students have learning fractions a number of recurring themes on remediation have emerged. Firstly, children bring informal knowledge to class and learning should build on this strength as much as possible (Mack 1990, 1995). For example, fair-sharing contexts offer a rich environment for introducing fractions and partitioning activities should be given to children from the beginning and continue through to the middle grades (Empson, 2002; Flores & Klein, 2005; Fosnot & Dolk, 2002; Sharp et al. 2002). There are recommendations to include mixed numbers and fractions greater than 1 from the start, while avoiding the use of the term improper fraction (Kamii & Clark, 1995; Van de Walle et al, 2011). Posing problems that draw out a variety of student-generated strategies and representations should be the core of fraction instruction (Empson, 2002, Flores & Klein, 2005; Sharp et al., 2002). Working with benchmark fractions can help students learn fraction number sense (Huinker, 1998; Reys et al., 1999). Opportunities to utilize a variety of different visual representations of fractions should be provided to children (Cramer & Henry, 2002; Reys et al. 1999; Van de Walle et al., 2011). Manipulatives and models have also been discussed as potential solutions to the problems, with mixed reviews. I single these out for detailed discussion as our understanding of their use and worth has been refined since the early days of reform.

Manipulatives

It is commonly agreed, even taken for granted, that the use of manipulatives is an essential component in good elementary mathematics instruction (Ball, 1992; Thompson, 2002).

However, researchers have found that there is no guarantee that the use of manipulatives will have the desired effect on students' learning (Baroody, 2002; Clements & McMillen, 2002; Thompson, 2002). A widely held assumption is that mathematics is embodied and evident in the physical materials themselves, but this is simply not the case: mathematical ideas are constructed in each human mind (Clements & McMillen, 2002). Each situation is open to interpretation and the student may not see what the teacher sees when looking at the same item (Ball, 1992; Thompson, 2002; von Glasersfeld, 2005). Another fallacy is the "notion that understanding comes through the fingertips" (Ball, 1992, p. 17), as if by physically manipulating objects students will learn mathematical concepts. Instead, many researchers believe that the learning that happens through the use of manipulatives is most successful when students reflect on their actions with the manipulatives, thereby linking ideas and making connections (Clements & McMillen, 2002; Stein & Bovalino, 2001; Thompson, 2002). The students' learning experience must be guided, but not too restricted, lest the activity become a matter of mindlessly following a procedure (Baroody, 2002; Stein & Bovalino, 2001).

Teachers need support in using manipulatives in this way in their classrooms (Ball, 1992). Those teachers who are trained in the use of manipulatives and have prepared well enjoy the most successful lessons (Stein & Bovalino, 2001). When planning a lesson, it is best if teachers consider what they want their students to understand, not what they want the students to do (Thompson, 2002). One of the most important aspects to consider is "that the experience be meaningful to students and that they become actively engaged in thinking about it" (Clements & McMillen, 2002, p. 260). A final point to take into consideration is that "certain computer manipulatives may be more beneficial than any physical manipulative" (Clements & McMillen, 2002, p. 260) for a variety of reasons.

Perhaps the reason that manipulatives do not work the magic that educators had hoped for can be found by exploring three different kinds of knowledge. Kamii and Warrington (1999) discussed the teaching and learning of fractions in light of Piaget's three types of knowledge: *social* (or conventional) *knowledge*, *physical knowledge*, and *logicomathematical knowledge*. The traditional method of teaching can be categorized as transmitting social knowledge, since the intent of the process is to pass the knowledge of algorithms from teacher to student. Manipulatives were recommended as a remedy for the shortcomings of traditional teaching methods with the hope that they would increase conceptual learning (Kamii & Warrington, 1999). As has been discussed, having students work with manipulatives, which is building physical knowledge, does not necessarily achieve the desired outcome. The thing to do is to focus on the development of reasoning, or logicomathematical knowledge, which "develops out of children's own mental actions" (Kamii & Warrington, 1999, pp. 85-86). Manipulatives are one of many types of models. What models have been recommended for learning about fractions?

Models for Fractions

Van de Walle (2007) gave the following definition of a model: a "*model for a mathematical concept* refers to any object, picture, or drawing that represents the concept or onto which the relationship for that concept can be imposed" (p. 31). Visual and physical models of fractional parts and wholes "can help students clarify ideas that are often confused in a purely symbolic mode" (p. 295). Van de Walle described three main categories of models for fractions: region or area models, measurement or length models, and quantity or set models. Area models encompass circular pies, rectangular regions, geoboards, grids or dot paper, pattern blocks, and paper folding. Length models include fraction strips, Cuisenaire rods, line segment drawings, number lines, and folded paper strips. Set models are groups of items such as counters or drawings of sets. It is important to use a variety of models — the same activity with different models will be different from the students' perspective (Van de Walle, 2007), although experts (e.g., teachers) "seem to view all modes of presentation of information as equivalent" (Bright, Behr, Post & Wachsmuth, 1988, p. 230).

Circle models are used most commonly (Van de Walle, 2007), but the use of circle models alone may unintentionally reinforce whole number thinking (Moss & Case, 1999). On the other hand, there has been great success with use of the circle model in fostering fraction number sense, as outlined above (Cramer & Henry, 2002; Cramer et al., 2002). Kamii and Clark (1995) hold a contrasting view, however, stating that "we would not provide any fraction circles and encourage children, instead, to make their own drawings" (p. 376). In this way, the drawings represent the children's own knowledge and understanding, unlike drawings "presented in textbooks, which represent someone else's thinking" (p. 376).

The use of the number line, another model in teaching fractions, is encouraged by many researchers and is connected to the real-world context of measuring (Van de Walle et al., 2011). After completing clinical and large-group teaching experiments with fourth and fifth graders on the subject of representing fractions and ordering fractions on number lines, Bright et al. wrote, "number line instruction is difficult" (1988, p. 227). Students struggled with representing equivalent fractions on a number line, did not successfully transfer knowledge to slightly different situations, and appeared to have had trouble connecting symbolic and pictorial information. It may be that the way the number line was introduced to these students was ineffective—it was presented to the students rather than developed with them in a meaningful context.

The experimental curriculum created by Moss and Case (1999) was aimed at helping children in Grade 4 develop concepts of rational numbers overall, not fractions alone. The curriculum was built on a linear measurement model they refer to as the number ribbon, which emphasized continuous quantity, measurement, and proportion rather than discontinuous quantity and counting. Moss and Case encouraged the children to work with landmark percent, decimal, and fraction equivalences. The program began with a beaker of water as a visual prop; the height of the beaker was the number ribbon model. In a question where the beaker was to hold a specific amount of water, the number ribbon functioned as a double number line, double because there are two numbers to compare: percentages and millilitres. Fosnot and Dolk (2002) developed the use of the double number line for the purpose of addition and subtraction of fractions with unlike denominators, as well as with percentages. It is interesting to note that Moss and Case (1999) witnessed their students spontaneously using a money model in the context of calculating percentages; Fosnot and Dolk (2002) also identified the money model and its usefulness for developing landmark decimals.

Another model proposed by Fosnot and Dolk (2002) is the clock model. They suggested it would be helpful for "adding and subtracting fractions like fourths, thirds, sixths, and twelfths" (p. 89). The clock model functions as a double number line by allowing common fractions to be converted into whole numbers of minutes, operated on, and then converted back into fractions. Chick et al. (2007) reported on using a clock face "as a way for students to think about fractional parts of a whole – as twelfths and their equivalent fractions" (p. 52). Two students in Chick's grade five class were given worksheets and were shown how to draw line segments from the centre of the clock face to the numerals 1 to 12 and to draw arcs to represent the motion of the minute hand. Samples of the work of the two children showed that they shaded in the sections when adding fractions, which is using the clock face as an area model. The children did not convert the fractions of an hour into minutes to solve problems. Chick et al. (2007) reported that close observation of two students as they worked with clock fractions revealed differences in the students' understanding. One student did not appear to have a conceptual understanding of fractions. She relied on memorized procedures for dealing with fractions, had difficulty identifying parts of the whole on diagrams of the clock face, and did not appear to understand fractions as relationships. The other student had a conceptual understanding of fractions which was revealed in the diagrams and symbols she produced, but interestingly had not been demonstrated on previous tests involving computations. Although both Fosnot and Dolk (2002) and Chick et al. (2007) made use of a clock model, they did so in very different ways that render these, in essence, different models: one a linear model based on the double number line and the other a circular area model.

The mixed reviews of the effectiveness of each of these models may be related to the context in which the model was used (particular models may work better to model particular problems), and also, how the model was introduced — was it given by the teacher or developed with the students?

The Activity of Modeling

Researchers have recently begun to carefully explore the issue of model introduction in mathematical lessons. In this view of mathematical models, "the use of pre-designed models is replaced by the activity of modeling" and "modeling is primarily seen as a form of organizing, within which both the symbolic means and the model itself emerge" (Gravemeijer, 2002, p. 7). Similarly, Fosnot and Dolk (2002) described mathematical models as "mental maps of relationships that can be used as tools when solving problems" (p. 90). In their view, the effective introduction and use of models has three stages. Models begin as the child's representations of action in a situation, develop into representations of the situation, and finally mature into a symbolic representation of *mathematizing*[3] (Fosnot & Dolk, 2002). Fosnot and Dolk insist that models must be developed within a rich context and learners must construct models for themselves; models cannot be transmitted (Fosnot & Dolk, 2002).

A practical question that arises out of this view of models and modeling is how does one teach students to construct models for themselves? Van Dijk and colleagues (van Dijk, van Oers & Terwel, 2003; van Dijk, van Oers, Terwel & van den Eeden, 2003) studied and compared two differing approaches; (a) providing ready-made models to students, and (b) guiding students in the co-construction of models. Qualitative observations showed that a child who was provided with models had difficulty using them, whereas a child who co-designed her own models had no difficulty making use of them (van Dijk, van Oers & Terwel, 2003). In a larger study, children who were taught to design models outperformed children who were directly provided models (van Dijk, van Oers, Terwel & van den Eeden, 2003). Researchers have investigated different problems, contexts, and calculations which promote children's creation of models as a means of solution. I chose to base the intervention in my case study on a *minilesson*[4] recounted Fosnot and Dolk (2002). The minilesson served as the foundation for the co-construction of the clock model. In the minilesson described by Fosnot and Dolk, the teacher posed a carefully designed series of fraction addition problems, embedded in a context that involved the use of time, with the purpose of eliciting responses from the children that would assist them in developing the use of the clock model.

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[3] Mathematizing means "interpreting, organizing, inquiring about, and constructing meaning through a mathematical lens" (Fosnot & Dolk, 2002, p. 18).
[4] A minilesson is about ten minutes long at the beginning of the math period and may be used "to highlight a computational strategy, share a problem-solving approach, or do mental math work" (Fosnot, 2007, p. 31).

Summary and Implications

There is no question that models are crucial for solving problems. Some authors contend that effective models are those that children construct for themselves (Fosnot & Dolk, 2002). If children can co-construct a model as a class together with the teacher, it is more beneficial than attempting to transmit the model directly from the teacher to the student (van Dijk, van Oers, & Terwel, 2003; van Dijk, van Oers, Terwel & van den Eeden, 2003). If these more recent theories about the effective development of mathematical models are applied with a specific, less-studied, model in the instruction of fractions — the clock model — what kind of impact would it have on student understanding of, and facility with, this often challenging area of mathematics? Knowing about this will help mathematics educators to focus fraction instruction time more effectively. Teachers want to select the most effective models to co-develop with their students.

CHAPTER THREE

Methodology

Research Design

This research was a qualitative case study examining the effect of the co-construction of the clock model on students' understanding of fractions in the context of reform mathematics instruction. As such, it was a single *instrumental case study* (Stake, 1995), because in-depth understanding of the issue was illustrated with one bounded case: the teacher and students in a class of Grade 4/5 students. A pretest, instruction, midtest, intervention, posttest, retention test sequence was employed. This research methodology is similar to Garrett (2010) who investigated the effectiveness of the reform method of teaching for helping Grade 7 students understand the concept of area. Garret used a pretest, intervention, posttest model in her case study.

Research Sample

The project was carried out with a *purposeful sample* (Creswell, 2007). The particular class was selected because the classroom teacher is devoted to teaching mathematics consistent with the principles of reform instruction. The research was conducted with a Grade 4/5 class in an urban elementary school in a Separate School Board in Northwestern Ontario. There were 23 students in the class, 11 students in Grade 4 and 12 students in Grade 5. Eleven students were female; 12 were male. Two students were on modified programs for mathematics. The class was predominantly white with several Aboriginal students. The socio-economic status was mixed.

Procedure

Ethics approval was obtained from Lakehead University, the school board, and the principal of the school where the study was conducted. Since student data were used, introductory letters and permission forms were sent to parents and guardians (see Appendices A and B); students received their own introductory letters and consent forms (see Appendices C and D). The teacher received her own introductory letter and consent form (see Appendices E and F). The school principal also received an introductory letter and consent form (see Appendices G and H). In order to conceal the identities of the board, school, teacher, and students, names have not been used and all associated wording in appendices has been altered. All parents and students gave consent for participation in the study.

The unit on fractions was taught over a period of about a month, beginning in the second week of April, 2011 and ending in the second week of May, 2011. The retention test was given six weeks after the end of the unit, at the end of June. Generally, the math period was about 90 minutes in length, and was scheduled in the late morning before the lunch break. Prior to the beginning of the study, I developed the pretest and fraction unit teaching plan to address six areas of student difficulty with fractions identified in the literature (see Tables 1 and 2, respectively). This unit plan was developed with the understanding that it would be modified by the classroom teacher, in consultation with me, to suit the needs of the class.

Table 1

Instrument Design

Letter	Difficulty cited in the literature	Citation	Instrument item	
			Pretest	Midtest
A	Fractional parts are equal-size portions	Pothier & Sawada (1983); Reys et. al (1999)	1	
B	Understanding fraction symbols	Mack (1995); Van de Walle (2007)	2	
C	Improper fractions	Mack (1990); Tzur (1999)	3	
D	Size of unit fractions	Mack (1990); Van de Walle (2007)	4	2
E	Estimation and comparison: fraction number sense	Cramer & Henry (2002); Cramer et al. (2002); Reys et al. (1999)	5, 6	
F	Equivalent fractions	Kamii & Clark (1995); Jigyel & Afamasaga-Fuata'i (2007)	7	1

Table 2

Fraction Unit Teaching Plan as Developed

Lesson	Topic	Difficulty addressed
Pretest		all
1	Fractions as parts of sets (Burns, 2001; Van de Walle et al., 2011)	B, E
2	Introduce fair-sharing with children's literature: *The Doorbell Rang* (Hutchins, 1986)	A, B, C, D, F
3	More fair sharing tasks in context, e.g., brownies, candy bars (Burns, 2001; Empson, 2002; Flores & Klein, 2005; Fosnot & Dolk, 2002; Sharp et al., 2002; Van de Walle et al., 2011)	A, B, C, D, F
4	Investigation (submarine sandwiches, adapted from Fosnot & Dolk, 2002)	A, B, D, E, F
5	Finish and take up Investigation in a congress[5] (Fosnot & Dolk, 2002)	A, B, D, E, F
6	Make rectangular fraction kits (Burns, 2001)	A, B, D, F
7	Play Cover-up and Uncover games with rectangular fraction kits (Burns, 2001)	A, B, D, F
8	Discuss games, play Uncover with an altered rule, do Cover the Whole (Burns, 2001)	A, B, D, F
9	Drawing fractional parts of sets (Burns, 2001; Van de Walle et al., 2011)	A, B, F
10	Fraction benchmarks (Burns 2001; Reys et al., 1999; Van de Walle et. al 2011)	B, E, F
11	Ordering fractions (Burns, 2001)	B, C, D, E, F
12	Adding and subtracting fractions (Burns, 2003)	B, C, F
Midtest		all
14	Minilesson: Addition using landmark fractions of an hour, The clock B1 (Imm, Fosnot & Uittenbogaard, 2007); Halving squares (Burns, 2001)	A, B, F
15	Minilesson: Addition using landmark fractions of an hour, The clock B2 (Imm et al., 2007); Exploring fractions with pattern blocks (Burns, 2001)	A, B, D, F
16	Minilesson: Addition using landmark fractions of an hour, The clock B3 (Imm et al., 2007); Wipeout game (Burns, 2001)	A, B, D, F
17	Minilesson: Subtraction using landmark fractions of an hour, The clock B7 (Imm et al., 2007); How much is blue? A pattern block activity (Burns, 2001)	A, B, D, F

[5] The congress is "a forum in which children communicate their ideas, solutions, problems, proofs, and conjectures to each other" (Fosnot, 2007, p. 27). The teacher plans and facilitates the congress.

18	Minilesson: Subtraction using landmark fractions of an hour, The clock B8 (Imm et al., 2007); Problems with oranges (Burns, 2003)	B, C, E, F
19	Minilesson: Mixed addition and subtraction using landmark fractions of an hour, The clock B9 (Imm et al., 2007); Fraction Capture Game (Burns, 2003)	A, B, D, E, F
20	Minilesson: Addition of fractions using equivalence, The Double Number Line (Imm et al., 2007); Fraction Capture Game, Version 2 (Burns, 2003)	A, B, D, E, F
21	Minilesson: Addition of fractions using equivalence, The Double Number Line (Imm et al., 2007); Balloons and brownies (Burns, 2003)	A, B, C, D
Posttest		all
Retention test	6 weeks after the Posttest	all

Table 3

Fraction Unit Teaching Plan as Implemented

Lesson	Topic	Difficulty addressed
Pretest		all
1	Fractions as parts of sets (Burns, 2001; Van de Walle et al., 2011)	B, E
2	Introduce fair-sharing with children's literature: *The Doorbell Rang* (Hutchins, 1986)	A, B, C, D, F
3	More fair sharing tasks in context, e.g., brownies, candy bars (Burns, 2001; Empson, 2002; Flores & Klein, 2005; Fosnot & Dolk, 2002; Sharp et al., 2002; Van de Walle et al., 2011)	A, B, C, D, F
4	Investigation (submarine sandwiches, adapted from Fosnot & Dolk, 2002)	A, B, D, E, F
5	Finish and take up investigation in a congress (Fosnot & Dolk, 2002)	A, B, D, E, F
6	Make rectangular fraction kits (Burns, 2001)	A, B, D, F
7	Play Cover-up and Uncover games with rectangular fraction kits (Burns, 2001)	A, B, D, F
8	Discuss games, play Uncover with an altered rule, do Cover the Whole (Burns, 2001)	A, B, D, F
9	Fraction games with supply teacher (no data collected)	A,B, D, F
10	Fraction benchmarks (Burns 2001; Reys et al., 1999; Van de Walle et. al 2011)	B, E, F
11	Drawing fractional parts of sets (Burns, 2001; Van de Walle et al., 2011)	A, B, F
12	Ordering fractions (Burns, 2001)	B, C, D, E, F
13	Review	all
Midtest		A, B, D, F
15	Minilesson: addition using landmark fractions of an hour, The clock B1 (Imm, Fosnot & Uittenbogaard, 2007)	A, B, D, F
16	Minilesson: addition using landmark fractions of an hour, The clock B2 (Imm et al., 2007)	A, B, D, F
17	Workout problem: investigation and congress	A, B, D, F
18	Three short problems, including one from Problems with oranges (Burns, 2003)	A, B, D, F
19	Bell work: review	all
Posttest		A, B, D, F
Retention test	Six weeks after the posttest	A, B, D, F

The unit began with the pretest (see Appendix I). Students were advised to do their best and that the results would assist in lesson planning for the unit. Students who required extra time to complete the test were able to do so during the lunch break that day. None of the tests were timed; students were allowed as much time as they needed to complete the tests.

The lessons were taught according to reform methods of instruction. The lessons were implemented by the classroom teacher based on the unit plan and took daily observations of the students into consideration as well as results of the pretest. Lessons 1 to 6 and 15 to 17 were videotaped to capture a record of the instruction. With parental permission, some students were videotaped or audio recorded as they solved problems together in class to provide a record of the development of their thinking. I recorded observation notes for each lesson and included matters such as how the lesson was taught, students' responses to problems, possible improvements, and direction for the next lesson. The first part of the fractions teaching unit included fractions as sets, fair-sharing cookies, the submarine sandwiches investigation, building the fraction kit and playing fraction games, benchmarks, and putting fractions in order, as outlined in Table 3.

The midtest (see Appendix J) was administered prior to the introduction of the clock model, in order to discern whether students had already developed the model for themselves and how familiar they were with the clock. Since there was concern that the pretest was too long for the students, adjustments were made and the later tests contained about half the number of questions.

The second part of the fractions teaching unit included minilessons designed to evoke the co-construction of the clock model and problems for students to solve. The four final lessons in the fractions unit are outlined in Table 3.

The unit ended with the posttest (see Appendix K). Six weeks after the end of the unit, the retention test (see Appendix L) was given. The post- and retention tests were very similar to the midtest, with the addition of one item from the pretest. These items were chosen after examining the results from the pre- and midtests and deciding what would be the most fruitful areas to examine, keeping in mind that a maximum of five questions were going to be selected. The results of the tests were not used for student evaluation, only for the purposes of the study, although the teacher had access to them for the purpose of informing instruction.

Data Collection

Consistent with case study methodology, data collection drew on multiple sources of information in order to build "an in-depth picture of the case" (Creswell, 2007, p. 132). The primary source of data was the assessments. During the fractions unit, some classes were videotaped in order to obtain high quality data about the instruction and student responses. In order to minimize disruption to the class, the video cameras were placed on tripods at the side and back of the room. The remote zoom feature allowed a closer look at how the students were interacting and solving problems together and at how the teacher interacted with them. The purpose of the videotaping was two-fold: first, to help answer the research question and second, to have available for later professional development for in-service teachers. In addition, audio recordings were made of whole class discussions and three pairs of students working together. The audio recorders were placed on bookshelves, desks, or the floor near where students were working. I reviewed the video and audio recordings daily and made notes, but did not analyse them in detail. Some student work completed in class was also collected, photocopied, and returned to students the following class. The teacher and I reviewed the students' written work for planning purposes. A further source of data was my daily observation journal entries, which contained my personal view of the lesson taught that day and student responses to the lessons. These observations provided me with insight into how the students were learning on a daily basis and how they were making use of the clock model in solving problems.

Data Analysis

Data were described, classified, interpreted, and represented (Creswell, 2007) in the following way. I coded all responses to test items in ATLAS.ti, a software program designed to assist researchers when analyzing qualitative data. The responses were coded as either correct or incorrect and also by type of solution strategy, model used, and/or big idea addressed. The a priori categories were based on *The landscape of learning: fractions, decimals, and percents*, which was first developed by Fosnot and Dolk (2002) and then updated and republished in 2007 (see Figure 1). Additional codes were developed from the literature and some codes emerged from the data. The code list can be found in Appendix M. Once all test responses were coded, the solutions were re-examined to look for evidence of the use of the clock model. A summary of data sources and analysis is portrayed in Table 4.

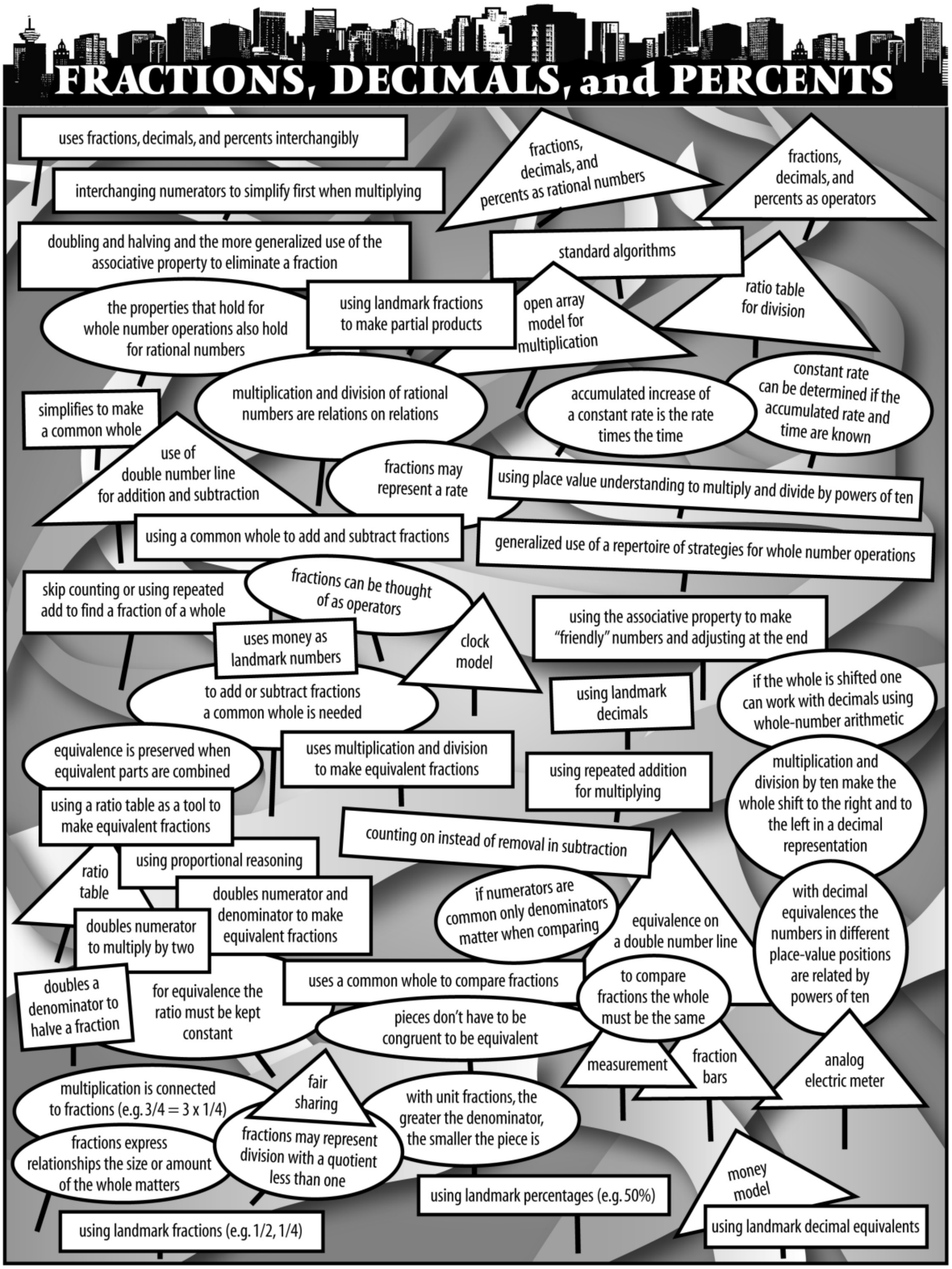

Figure 1. The landscape of learning: fractions, decimals, and percents on the horizon showing landmark strategies (rectangles), big ideas (ovals), and models (triangles). (Fosnot 2007, p. 15)

Table 4

Summary of Data Sources and Analyses

Data source	Type of analysis
Pre-, mid-, post-, and retention test responses	Coded correct/incorrect/no response and for model, strategy, and/or big idea.
Video and audio recordings	Reviewed daily to inform instruction and to analyze role of the teacher and student use of models in solving problems
Photocopied student work	Reviewed to inform instruction
Observation journal	Reviewed daily to inform instruction and to analyze role of the teacher and student use of models in solving problems

CHAPTER FOUR

Findings

The impact of the co-construction and use of the clock model on students' understanding of fractions was assessed by the evaluation of student responses to fraction problems. The responses to each pre-, mid-, post-, and retention test question were analysed and coded. A summary of the sources of student responses is shown in Table 5.

Table 5

Sources of Data

Test	No. Students	No. Questions	No. Primary Documents[6]
Pretest	22	9	198
Midtest	23	4	92
Posttest	22	5	110
Retention Test	22	5	110

Section 1: Results of the Pretest and Analysis

The classroom teacher administered the pretest. The purpose of the pretest was to get a sense of the students' understanding before beginning the fractions unit. In this section I present a summary of the results of the class and the principal findings. The nine items were grouped into four types of items for analysis: compare, fractions as division, meaning of fractional notation, and addition. A discussion follows after Table 6, which contains a summary showing the percentages[7] of correct and incorrect responses.

[6] Primary Documents refer to documents within the ATLAS.ti program. Each student solution to a problem is a separate document.
[7] Results are shown as percentages rather than numbers of students for ease of comparison as not all students wrote all of the tests resulting in slightly different *N*s.

Table 6

Results of Pretest (N = 22)

Question Number and Short Form	Question Wording	Correct %	Incorrect %	No Response %
COMPARE				
7 Compare Area	Are the following shaded circles equal? Compare and explain your answer. (circle models provided) (Jigyel & Afamasaga-Fuata'i, 2007, p. 21)	95	5	0
5 Pizza	Joey and Robert each had the same size pizza. Joey cut his pizza into eight equal size pieces and ate six of them. Robert cut his pizza into five equal size pieces and ate four of them. Who ate more pizza? (Burns, 2001, p. 138)	36	64	0
FRACTIONS AS DIVISION				
1 Divide Brownies	4 children are sharing 10 brownies so that everyone gets the same amount. How much brownie can 1 person have? (Empson, 2002, pp. 29-30)	41	59	0
MEANING of FRACTIONAL NOTATION				
2 Meaning	What does the bottom number in a fraction tell us? What does the top number in a fraction tell us? (Van de Walle, 2007, p. 299)	27	41	32
4 Order	Put these fractions in order from smallest to largest: $^1/_5$, $^1/_3$, and $^1/_4$ (Cramer & Henry, 2002, p. 43)	23	77	0
ADDITION				
9 Exercise (Landmarks) Add	Joel worked out at the Complex the other day. He ran for half of an hour and then walked for $^1/_4$ of an hour. How much of the hour did he spend exercising? (adapted from Imm et al., 2007, p. 27)	27	64	9
6 Estimate Add	Is the following sum larger or smaller than 1? Explain how you know. Use estimation. $^1/_2 + {}^1/_3$ (Reys et al., 1999, p. 530)	18	73	9
8 Add twelfths	Add. $^3/_4 + {}^2/_{12}$ (Imm et al., 2007, p. 24)	14	82	5
3 Add pies	If Linda ate one half of an apple pie and two thirds of a cherry pie, how much did she eat? (adapted from Sharp et al., 2002, p. 21)	5	91	5

Comparing fractions

Compare area. Students had the most success with Pretest Question 7 (95% correct), which involved comparing the two fractions $^{2}/_{3}$ and $^{4}/_{6}$ represented in visual form by the shaded areas of two circles. Students in the case study were able to answer this question correctly, although they usually lacked the right vocabulary; for example, one student wrote, "Yes they are because 4 qurters and 2 therrds are the same" (*sic*) (P141)[8]. Only one student wrote an answer that did not make sense (P137). This is different from the results of Jigyel and Afamasaga-Fuata'i (2007), who found a variety of correct and incorrect explanations. Correct explanations in their study included the ideas of equal areas and equivalent fractions. With incorrect explanations, students viewed fractions as two unrelated whole numbers so more pieces meant more area. Students in the case study did not think that more pieces meant more area. By the end of Grade 3 in Ontario, students are expected to "identify congruent two-dimensional shapes by manipulating and matching concrete materials" (MOE, 2005, p. 59) so prior experiences with comparing shapes may be the reason why most students in the case study answered this question with ease.

Divide and compare pizza. Pretest Question 5 (Pizza) was chosen to find out information about the students' ability to compare fractions in a context. Only one third (eight) of the students who were able to compare the shaded area fractions in the first item, were able to answer this question correctly. It required that they use a common whole, accurately partitioning and comparing areas (e.g., P95). Students did not necessarily express the amounts of pizza as fractions. They were more likely to simply draw a picture to represent the situation and try to figure it out by physically comparing the areas or counting the numbers of pieces. The use of a common denominator for comparing eighths and fifths, namely fortieths, was far beyond the

[8] P means Primary Document within the ATLAS.ti program. The number indicates the specific document.

students in this group. Many students were not able to draw fifths accurately—even some of the students who nonetheless answered the question correctly. This is dissimilar to Burns (2001) who found that students primarily used one of two strategies: students made drawings, used fraction symbols to express the quantities of pizza, and either compared how much pizza was left or rewrote the fractions so that they had common denominators and compared the amount of pizza eaten. It is probable that students in the case study had had less instruction than those in Burns' cohort, especially since this was a pretest item and Burns described typical results when this problem was used as an assessment after instruction.

An important big idea when comparing fractions is the whole must be the same (Fosnot, 2007), which is why the question specifies that the boys began with the same size pizza. In the drawings made by the students in the class, only four students had represented the pizzas as being about the same size. Although it is possible to compare pizzas of different sizes by thinking about the proportions of the pieces, it is more likely that the students were not attuned to the importance of this big idea. Students who said that Joey ate more pizza may have only been considering the number of pieces, not yet having constructed the big idea that "fractions express relationships; the size or amount of the whole matters" (Fosnot, 2007, p. 15).

Fractions as fair sharing division

Pretest Question 1 (Divide Brownies) was chosen to find out how the students would solve a fair-sharing problem. Would students completely partition the brownies into equal-size portions? Not surprisingly, the results are very similar to the pizza sharing item. The two most common responses were $2^1/_2$ brownies each, given by 41% of the class (nine students), and 2 brownies each, given by 36% (eight students). Some students mentioned the two leftover brownies; some did not. When fraction understanding is more fully developed, one would expect the students to fair-share all of the brownies because, true to the context of the question, brownies can easily be cut into portions. The results are similar to Empson's (2002) description of the development of strategies for fair sharing. Empson explained that almost all children have initial invented strategies for fair sharing which can be developed through teacher questioning. For example, a teacher can ask a child if there is anything they could do to share the last two brownies.

A key beginning strategy is "using landmark fractions" (Fosnot, 2007, p. 15) which can be developed by solving fair-sharing problems such as this one. Seven students, about one third of the class, successfully used this strategy to answer the problem. Four students displayed a limited understanding of the question: their responses showed that they did not interpret the question correctly. In summary, less than half of the class could fair share or use landmark fractions in a context without additional prompting.

Understanding fraction symbols

Fraction notation. The responses to Question 2 (Meaning) taken together with Question 4 (Order) revealed that students were generally neither comfortable, nor familiar, with symbolic fraction notation. Only six students, approximately 25% of the class, were able to give a reasonable explanation of fraction notation in Question 2 (Meaning).

Fraction notation size. Pretest Question 4 (Order) asked students to put these fractions in order from smallest to largest: $^1/_5$, $^1/_3$, and $^1/_4$. Putting fractions in the correct order requires understanding the sizes of the fractions and the relationship between the numerator and denominator. Only 23% of the class (five students) ordered the fractions correctly. This demonstrates that, at the time of the pretest, most students had not constructed the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p. 15). Many of the students used their knowledge of whole numbers to order the fractions instead of thinking about the meanings of the fractions. Also, many of the responses demonstrated that the students had a limited understanding of fraction symbols. For example, a few students separated the numerators and denominators and made a list from smallest to largest similar to this: 1, 1, 1, 3, 4, 5 (P81).

Adding fractions

Addition exercise. Pretest Question 9 (Exercise - Landmarks) was asked to see if students had a strategy of their own to add fractions in a context that could evoke the use of the clock model. Half an hour and one quarter of an hour were chosen since they are common landmarks used in everyday situations. Six students (27% of the class) were able to provide a reasonable answer to the question, either 45 minutes (four students) or ¾ of an hour (two students). One of these students drew a circle, titled it "the hour", divided it into fourths, shaded three fourths, and responded with ¾ of the hour (P194). Another student drew a clock face and faintly drew in three lines, like the minute hand, from the centre to 12, 6 and 9. Her response was 45 minutes (P183). There was a wide variety of incorrect answers. Of these, six showed some understanding of the relationship between one half and thirty minutes or one quarter and fifteen minutes. Seven students had a very limited understanding and seemed unable to interpret the question. I think some of them did not read the phrase "half an hour" and thought it was a whole hour or just totally ignored that part of the question.

Estimate addition. Pretest Question 6 was chosen to document the students' initial ability to estimate using fractions, which requires having fraction number sense. This proved to be more difficult than the addition of benchmark fractions as just one quarter of the students were able to solve the problem. Reys et al. (1999) found that about one third of the students interviewed said that since $^1/_3$ is less than $^1/_2$, the sum must be less than 1. Only one student in the group explained it this way (P112). Three others had another correct explanation as to why the sum would be less than 1: they used drawings that were close to being accurate. Two drew circles and showed that the total was around ¾, clearly less than one whole. The other looked at the leftover part, and called it one eighth. Their drawings were somewhat inaccurate, but the answers were still correct. Two students were able to give the correct answer but their justification was not correct: they shaded $^1/_2$ of a circle and then proceeded to shade $^1/_3$ of the other half, actually $^1/_6$ of the whole. In general, students had trouble interpreting the fraction notation, were often not able to represent one third properly in their drawings, and thought of $^1/_3$ as larger than $^1/_2$, probably due to thinking about whole number relationships, as in the responses to Question 4 (Order). I believe that some children did not understand the question. Two students left it blank. Perhaps they did not know what $^1/_3$ was or how to represent it with a drawing, or did not understand the two-part question about the sum being less than or greater than 1. Another possibility is that they did not know the meaning of the term "sum."

Bare[9] addition. Pretest Question 8 ($^3/_4 + ^2/_{12}$), was asked to see if students already had a strategy for adding fractions that could be added using the clock model. Only 14% of the class (three students) was able to answer the question correctly. Two students were able to draw twelfths on a circle or circles and work out an answer expressed as $^{11}/_{12}$. A third student called her result "One pie with out one 12th" (*sic*) (P165). One student had the correct answer represented on a circle diagram, but did not write the answer in numbers (P162). Four other students were able to represent the fractions on circles but were unable to combine. There were two main types of natural number or "N-distractor" errors (Streefland, 1993, p. 300). Streefland explained that when students use the arithmetic rules of natural numbers to operate on fractions,

[9] Bare questions consist of numbers only. Context questions are story questions or word problems.

it indicates that they do not have an adequate understanding of fractions. First, five students simply added across the fractions to get $^5/_{16}$. Second, four other students added up the numbers to get either sums of 7 and 14 separately, or a total of 21. A third type of error was that one student added numerators and denominators and reported the answers separately. In each case, the students demonstrated that they were not considering fractions as relationships because they treated the numbers separately.

Add pies. Just one student was able to solve Pretest Question 3 (5% correct). This question was chosen to see if students could deal with combining fractions $^1/_2 + {}^2/_3$ to get a result greater than 1 whole. Instead of getting information about their ability to work with improper fractions, what I found was that many students were unable to deal with the two thirds properly. They often drew fourths instead of thirds or, if they were able to represent them correctly with diagrams, were unable to find a way to combine the half and two thirds together. A question using one half and three fourths may have been more suitable at this stage. Nevertheless, Sharp et al. (2002) reported results of using this question with a six year-old, who over time, had developed her own methods to "break them into same-size pieces" (p. 21). Only one of the students in this study was able to do this successfully at the time of the pretest (P51). She "over-marked" (Lamon, 1996); that is, made more partitions than were necessary to answer the question by making twelfths when only sixths were needed.

Summary

In summary, the pretest results show that, at the outset of the fractions unit, when questions were presented visually (Question 7) or in a context (Questions 5 and 9) students were more likely to respond with a correct answer than they were for a bare problem. The exception to this was Question 3 (Pies), which required working with thirds, which were problematic.

Students' lack of knowledge of fraction symbols was a factor which hindered their ability to respond to the questions. This is not surprising because, in addition to the fact that fraction notation is complex, the Ontario Curriculum states that students are not expected to use fraction notation by the end of Grade 3, only fractional names in words (MOE, 2005, p. 55). This means that the Grade 4 students may have had limited exposure to fractional notation.

Working with the lens of the *Landscape of learning: fractions, decimals and percent* (Fosnot, 2007), student solutions demonstrated that less than half the class used landmark fractions, a basic strategy at the very bottom of the *Landscape* (see Figure 1). Only one fourth of the class had constructed the big idea that "with unit fractions, the greater the denominator, the smaller the piece is" (p. 15).

Section 2: Classroom Instruction

I developed the initial fractions teaching unit plan based on the literature (see Table 2). The classroom teacher was already familiar with most of the literature and the lessons that I had selected for the core instruction, Lessons 1 to 12. We quickly developed a rapport during preliminary meetings prior to beginning of the study. This professional relationship allowed for rich discussions about student learning and a level of ease when either one of us was suggesting changes to the instruction. My role in the classroom was that of an observer. I made field notes and reviewed to the audio and video recordings at the end of the day. The teacher and I both looked at students' written work. The teacher and I made time to debrief after most lessons, either in person or by email later that day. On a daily basis, we refined the unit plan together to meet student needs (see Table 3).

The core fraction instruction included fractions as sets, fair-sharing cookies, the submarine sandwiches investigation, building the fraction kit and playing fraction games, benchmarks, and putting fractions in order. The specific details of these lessons can be found in the references provided in Table 3.

The approach used by the teacher included teaching through problem solving and promoting mathematical communication between and among students as well as with the teacher. The teacher posed open-ended problems which could be solved using a variety of strategies. The teacher did not tell or show the students how to solve the problems. Students often worked together in pairs, designated by the teacher, to solve problems. The teacher encouraged students to talk together while they solved problems. Whole class discussions were also orchestrated by the teacher, where she employed questioning techniques to deepen student learning. Students were asked to share, discuss, and defend their ideas.

Section 3: Results of the Midtest and Analysis

The classroom teacher reported that students struggled with the length of the pretest, so we decided the other tests would need to comprise fewer items. The questions were chosen to give students opportunities to demonstrate their use of strategies and models, and their construction of the big ideas, in the areas of comparison and addition. The purpose of the midtest was to get a sense of the students' understanding of fractions after the two week teaching unit and before the co-construction and use of the clock model. Twenty-three students wrote the midtest. In this section I present the findings and a summary of the results of the class.

Understanding fraction symbols

By examining all student responses for all of the midtest questions, there is clear evidence to say that 18 students, or 78% of the class, could interpret fraction symbols at the time of the midtest, whereas only 27% (six students) could in the pretest. For four of the students (17%), there was not enough information given in their answers. For example, one student only answered two questions on the midtest, and of the two responses, only one included a diagram (P271). He seemed to have a sense of the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p. 15) even though the rectangles he drew were out of proportion. However, this one response is not enough to claim that he understood fraction notation. Taken together with his work on the midtest and posttest, it is likely that his understanding was very limited. And lastly, one of the students clearly did not grasp the relationship between the numerator and denominator. He treated them as two separate numbers and his typical work included drawings of rectangles for each numerator and each denominator (P203).

Comparing fractions

Compare unit fractions. This question (Which fraction is larger, $^{1}/_{6}$ or $^{1}/_{8}$?) was selected in order to learn about the students' thinking with regard to the relative size of unit fractions. A similar question was asked on the pretest (Order $^{1}/_{5}$, $^{1}/_{3}$, and $^{1}/_{4}$) and 23% of students answered correctly. After two weeks of instruction, 18 students, 78% of the class, were now able to correctly answer the Compare question. Importantly, students were not able to use their fraction kits directly to answer the question, since they had made eighths but not sixths. Nine students drew some sort of rectangle model that resembled their kits; of those, seven responses were correct. Five students drew some version of circle models; of those, four were correct. For five of the students who correctly indicated which fraction was larger, either their written explanations were mixed up or their strategies were not shown. The errors made by students when answering this question included basing their reasoning on whole numbers rather than fractions and not using a common whole. Students who do not use a common whole are not reasoning proportionally. Several different appropriate written explanations were given, most commonly: “fewer pieces means pieces are larger” (four occurrences), “smaller denominator means larger pieces” (three occurrences), “greater denominator means smaller pieces” (two occurrences), and “more pieces means smaller pieces” (two occurrences). These results suggest that most of the students had now developed an understanding of the meaning of the fraction notation and had also constructed an important big idea about fractions, “with unit fractions, the greater the denominator, the smaller the piece is” (Fosnot, 2007, p. 15).

Equivalence. The purpose of the Equivalence question, “Does $^4/_6 = {}^2/_3$? Explain your answer.”, was to see how the students were thinking about equivalent fractions after the initial two weeks and what strategies they used. Fifteen students (65%) gave correct answers coupled with correct explanations. Thirds and sixths were not part of the fraction kits they had built, so this indicates they were able to visualize or represent the fractions in some way other than physically using their fraction kit in order to answer the question. Van de Walle et al. (2011) discussed four potential correct responses: two procedural responses and two conceptual explanations. According to Van de Walle et al., the essential concept is that “two fractions are equivalent if they are representations for the same amount or quantity—if they represent the same number” (2011, p. 310). For Fosnot (2007), the big idea is expressed as “for equivalence the ratio must be kept constant” (p. 15).

Jigyel and Afamasaga-Fuata’i (2007) asked students a similar question in their study; however, they specifically directed children to draw a model to represent the two fractions and then compare. In their case, almost all of the students represented the fractions correctly with area models, most commonly circles but also rectangles. Jigyel and Afamasaga-Fuata’i (2007) stated that their results suggested that representing thirds and sixth with diagrams was easy. Conversely, in my study I found that students had difficulty representing sixths with a circle model. Seven students drew circle models and answered the question correctly, although only two of those students drew the sixths properly. Ten students chose to draw rectangles. Of those, seven answered the question correctly. The main difficulty was that students did not use a common whole when making the comparison. Students used a variety of strategies, eight different strategies in total. The most commonly used strategy was matching areas, used by nine students in combination with drawing rectangles (five occurrences) or circles (four occurrences). Students who were able to model fractions for accurate comparison had begun to construct the big idea that "to compare fractions the whole must be the same" (Fosnot, 2007, p. 15).

Adding fractions

Exercise. Just four students (17%) worked out a reasonable answer to the Exercise question ($^1/_6 + ^1/_2$, in context); two wrote $^4/_6$ and two wrote 40 minutes. Each used a different strategy and/or model combination. Typical errors included N-distractor errors and not using a common whole when drawing rectangular models. It is of interest to note that, despite the context of time, only five students worked with or represented their answer in terms of minutes. Sixteen students used only fractions. Of the five who worked with minutes, the two who worked out a correct response had a way of figuring out that $^1/_6$ of an hour is ten minutes. The other three students were close, with answers of 35 or 45 minutes, because they thought that $^1/_6$ of an hour was either 5 minutes (poor transfer of circle diagram to clock face drawing) or 15 minutes.

Add sixths. This bare question, "Add. $^2/_3 + ^1/_6$", was chosen to see what strategies and models students would use to combine fractions when they were not able rely on their kits, and could potentially use the clock model as a tool for solving. Two models were used by the students: rectangles and circles. A total of nine students drew rectangular models. Of those, three students were able to draw reasonable diagrams but were not able to combine the fractions to give a final numerical answer. Only one student was able to correctly solve the problem using a rectangle model. Three students drew circle models; two were poorly done and only one student was able to represent the thirds and sixths reasonably well. He was not able to combine the fractions together to report a numerical answer. Four students simply added across numerators and denominators and gave the response $^3/_9$. They did not draw any diagrams and were not thinking about the meaning of the fractions.

Summary

On the midtest, 65-78% of the students were working at Ontario Grade 4/5 level expectations; that is, they were able to read, represent, compare and order simple (Grade 4) or proper (Grade 5) fractions (MOE, 2005). Addition of fractions is not expected until Grade 7 in Ontario. Most students had begun to construct the following big ideas: "fractions express relationships; the size or amount of the whole matters", "with unit fractions, the greater the denominator, the smaller the piece is", and "to compare fractions the whole must be the same" (Fosnot & Dolk, 2007, p.15). To represent fractions with drawings, the students used rectangles reminiscent of their fraction kits and circles. It is significant that students were able to make drawings of thirds and sixths with rectangles, since these pieces were not in their fraction kits. This means that students were using the model as *a tool for thought* (Fosnot & Dolk, 2002, p. 17).

Section 4: Description of the Intervention

The purpose of the intervention was to provide the opportunity for the students and teacher to co-construct the clock model. There were four math instructional periods between the midtest and the posttest. The focus during the first and second math periods was on *strings*[10]

[10] A string is "a structured series (a string!) of computations that are related in such a way as to develop and highlight number relationships and operations" (Fosnot & Dolk, 2002, p. 112)

that were designed to evoke the co-construction of the clock model. On the third day, a time context problem was posed; children investigated in pairs and the teacher facilitated a congress. Several short problems were posed on the fourth day. Outside of the math period, the teacher also had the children answer some fraction review questions in their workbooks as *Bell Work*, which was work they did first thing in the morning. Bell Work on fractions was done on the first day and also on the morning of the posttest.

The basis of the instruction on first day of the intervention was a minilesson which was described by Fosnot and colleagues (Cameron, Werner, Fosnot & Hersh, 2006; Fosnot & Dolk, 2002; Imm, Fosnot & Uittenbogaard, 2007). In consultation with me, the classroom teacher adapted the context of the minilesson so that it would be of interest to her students; she talked about her personal running program. Also, together we decided to begin the string with $^1/_2 + {}^1/_4$, rather than $^1/_2 + {}^1/_3$. This decision was made taking two aspects into consideration. First, the purpose of the string was to co-construct the clock model so we wanted to use the familiar landmark fractions that are common in daily usage. And second, the ages of the students were also taken into consideration because the students described by Fosnot and colleagues were in a higher grade.

Students sat on the carpet together in a corner of the classroom for the minilessons. The teacher engaged the students by discussing her running program and then posing questions to them. After posing each question, the teacher insisted on quiet time for students to think individually before asking students to share their thinking. There was much discussion by the students, moderated by the teacher, as they shared their thinking. The teacher scribed their thinking on the blackboard. This minilesson lasted about half an hour and consisted of three questions, all posed by the teacher in the context of her running program: $^{1}/_{2} + ^{1}/_{4}$, $^{1}/_{2} + ^{1}/_{3}$, and $^{1}/_{6} + ^{1}/_{2}$.

The teacher began the second minilesson by repeating one of the questions she had posed on the previous day, $^{1}/_{2} + ^{1}/_{3}$, particularly for the benefit of a couple of students who had been absent. Then she posed a new question, $^{1}/_{3} + ^{1}/_{4}$, grounded in the context of the class walking to and from a nearby building for field trip that morning. For the third question, the teacher returned to the context of her personal exercise program and asked $^{3}/_{4} + ^{1}/_{3}$. The context for the final question, $^{2}/_{3} + ^{1}/_{6}$, was personal for the children: they had walked outside for $^{2}/_{3}$ of an hour and played outside for $^{1}/_{6}$ of an hour. This minilesson lasted for about 25 minutes.

The teacher composed a problem for students to solve in pairs during the third math period of the intervention. The problem she posed was $^{1}/_{4} + ^{1}/_{6} + ^{7}/_{12}$: "Yesterday for my workout, I walked for a quarter of an hour, did sit-ups for a sixth of an hour, and ran for seven twelfths of an hour. How long was my workout?" Students worked on solving this problem and then the teacher facilitated a congress with the class. The math period was about 40 minutes long.

To begin the fourth math period of the intervention, the teacher put three of the student solutions to the workout problem from the previous day up on the board and they reviewed the student work together as a class. After that, students worked at their desks to solve three short fractions problems on a worksheet prepared by the teacher. The first question was about equivalent fractions; the second was a word problem in a time context; and the third was an orange problem adapted from Burns (2003).

Section 5: Results of the Posttest and Analysis

The posttest was written at the end of the fractions learning unit, after the co-construction and use of the clock model.

Understanding fraction symbols

By looking at all of the responses to the posttest questions there is evidence that 18 students (82%) could interpret fraction symbols. There were two students who clearly still did not understand and two students for whom it was unclear due to the fact that they did not use diagrams; they converted fractions into whole numbers of minutes and did not convert final answers back into fractions. This situation is similar to students who consistently convert fractions into decimals because they do not understand the relationship between the numerator and denominator.

Comparing fractions

Student ability to compare fractions increased slightly on the posttest over the midtest.

Compare unit fractions. For the Compare question, "Which fraction is larger, $^1/_6$ or $^1/_8$?", 86% (19 students) answered correctly. The most commonly used model was the circle. Six (27%) used a circle model and even if their sixths were drawn incorrectly, they had another strategy which they used in their written explanation. Because of the choice of fractions in this question, students were not able to directly use their fraction kits or the clock model: $^1/_8$ of an hour is equal to $7^1/_2$ minutes and was not used in class discussions. Four students (18%) successfully used a rectangle model to answer this question. Four (18%) more made use of a clock in some way. Two used it like an area model, one by drawing a clock face and dividing it into sixths as shown in Figure 2 (P294) and one by describing the cutting of a clock into six pieces and eight pieces as shown in Figure 3 (P297).

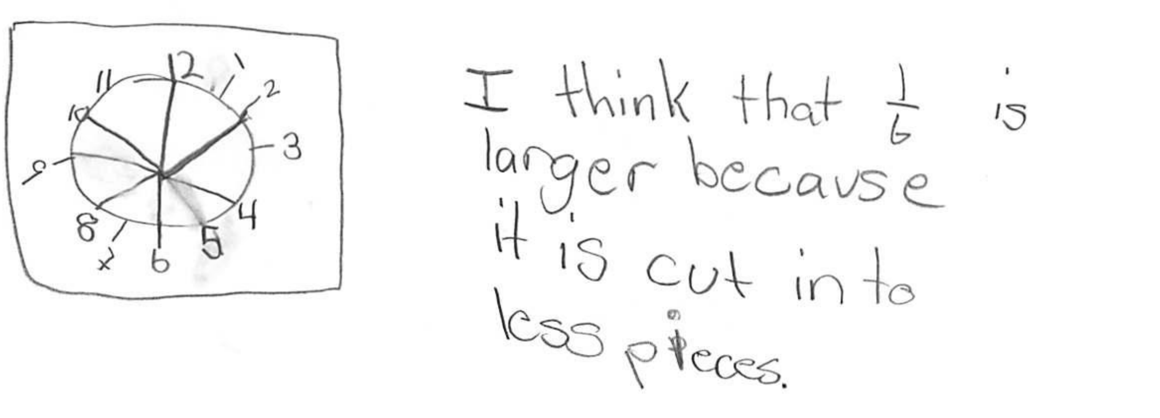

Figure 2. P294 Use of clock as area model for comparing unit fractions, drawing

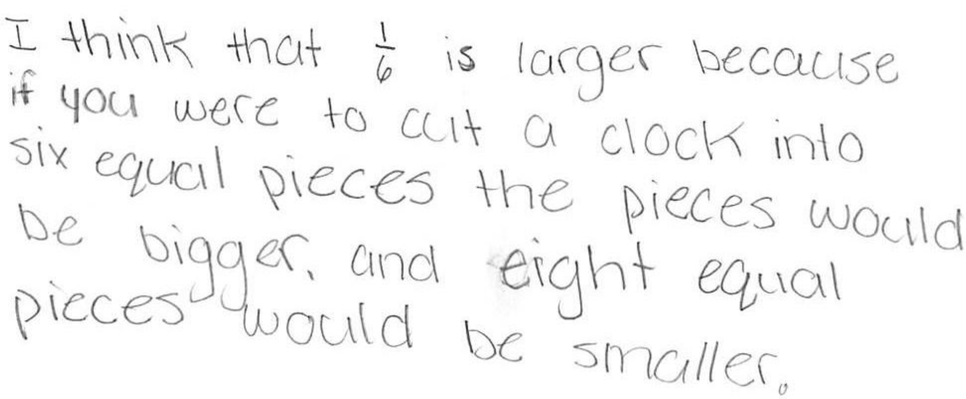

Figure 3. P297 Use of clock as area model for comparing unit fractions, description only

Two students referred to the clock and wrote that $^1/_6$ is equal to 10 minutes and although they did not have the correct number of minutes for $^1/_8$, they were able to reason the correct answer. Seven different justifications were written by the students. The most common explanations were "the smaller denominator means a larger piece" (seven occurrences) and "fewer pieces means that pieces are larger" (five occurrences). Three students wrote that $^1/_6$ is larger but their explanations were mixed up. I think this means they were in the process of constructing the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p.

15). Of the three who answered incorrectly, only one student seemed to be relying on the logic of whole numbers combined with an inaccurate circle drawing of sixths. Two students displayed limited understanding in their written responses. Most students therefore understood how a unit fraction is different from a whole number; that is, the meaning of $^{1}/_{8}$ rather than 8.

Equivalence. Nineteen students (86%) gave correct responses to the Equivalence question, "Does $^{4}/_{6} = {}^{2}/_{3}$?". Fourteen students used one or both of two main strategies in combination with drawing a diagram: matching the areas (eight occurrences) and referencing the unit fraction $^{1}/_{3} = {}^{2}/_{6}$ so $^{2}/_{3} = {}^{4}/_{6}$ (seven occurrences). In comparison, only two students referenced the unit fraction when answering this question on the midtest. It may be that students knew this relationship or they may have been verbalising the relationship they saw in their diagram. In one case, however, there was no diagram. This student may have visualized the relationship between the fractions or may have reached a level of *automaticity*. Automaticity refers to the ability of a student to quickly produce facts that have been committed to memory because the student has constructed relationships between the numbers (Fosnot & Dolk, 2001). There was an overlap in the use of strategies and models; students used more than one method to explain their thinking. For example, five students drew diagrams and indicated matching areas as well as referencing the unit fraction (e.g., P316). One student's rich response included drawings of circles and rectangles as well as an explanation which referenced the fractions as numbers of minutes. Four different models were used by the students: rectangle (seven occurrences), clock equivalence facts (six occurrences), circle with sixths drawn incorrectly (five occurrences), and clock model – area (two occurrences). Six students did not include any drawings with their solutions and either referenced a unit fraction (one student) or used clock equivalence facts (five students) to justify their answers. Most students therefore also understood that the same area can have different names.

Adding fractions

Exercise. Fifteen students (68%) worked out a correct answer for the Exercise question ($^1/_6 + {}^1/_2$, in context). Seven students (32%) used an "open clock" model, which is not discussed in the literature. I have described it in the section below on student use of the clock model. Two students used the clock area model to solve the problem; two students used clock facts with equivalent fractions; two students used clock facts in minutes. The most common problem leading to an incorrect answer was by students who thought $^1/_6$ of an hour was 15 minutes (two students). They were using clock facts in minutes but had the wrong number of minutes.

Add sixths. About half the students still found bare addition problems too difficult. A wide variety of correct and incorrect solutions were given for the Add Sixths question ($^2/_3 + {}^1/_6$, bare). Nine students (41%) gave correct answers, versus only 4% on the midtest. Three students used the open clock (e.g., P346), three used clock facts – fractions, one clearly used a rectangle model (P343), one made use of a circle model (P344), and one used the area clock model (P342). There were a variety of reasons students gave the wrong answer. Three students (13%) made some kind of an error using the clock as an area model, such as making a poor diagram or adding the wrong fractions together. An error made by three students was that they did not write their final answer as a fraction. Of these, only one would have been correct; the other two students had made additional errors. Two students were able to make drawings (one a rectangle and one a circle) but were not able to combine the fractions numerically. They understood what they needed to do but could not name it. One student used the open clock but added the wrong fractions together (P355). She added the unit fraction $^1/_3$ instead of the common fraction $^2/_3$, as shown in Figure 4. Only two students had N-distractor errors and two students did not respond to the question.

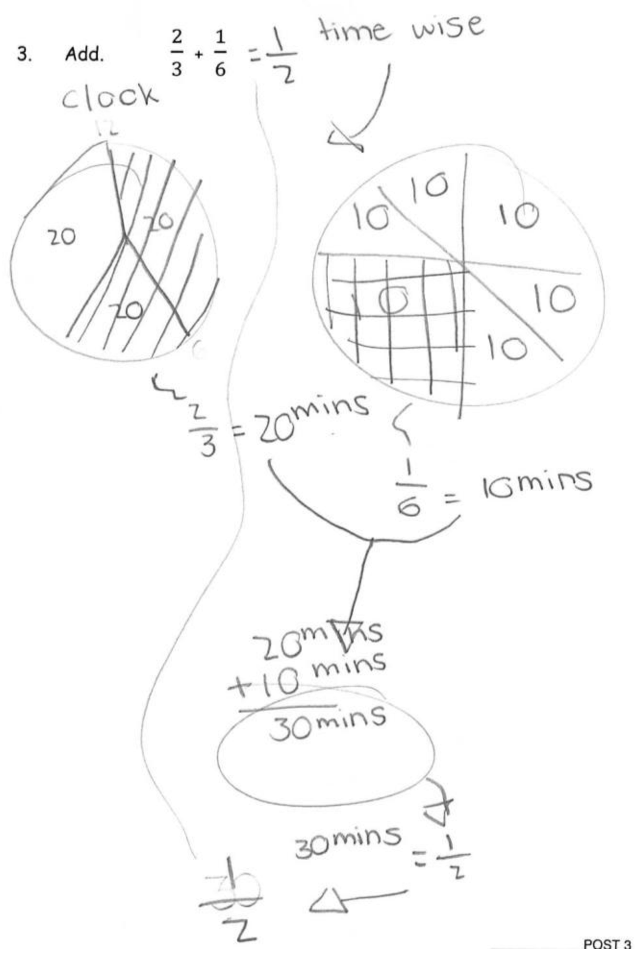

Figure 4. P355 Student added unit fraction instead of common fraction

Add twelfths. Eight students (36%) worked out a correct response to the Add Twelfths question ($^3/_4 + {}^2/_{12}$, bare). Although the students had made fourths as part of their fraction kit, they had not made twelfths. Their main experience with twelfths was working with the clock. Three students successfully solved the problem with clock equivalence facts in fraction form and three students used the open clock. One student used the area clock model and one used a rectangle model. Nine additional students (41%) attempted to use a version of the clock model, but were unsuccessful. The primary problem was that students who converted the fractions to minutes represented their final answer as a whole number, rather than as a fraction (five students). Three students used circle models and were either unable to combine the fractions (two students) or made a mistake (one student, P389). Two students used rectangle models; one reported a whole number answer of minutes and one was not able to make an accurate representation. This student also began and abandoned a clock model drawing. Two students added across numerators and denominators and one student did not answer the question.

Summary

The results of the posttest show that most students (82-86%) could successfully interpret fraction symbols, compare fractions, and make equivalent fractions. This meets the Ministry expectations for Grades 4 and 5. About half of the students were able to add fractions in context and one third solved the bare questions. In their written work, students often wrote rich responses which included a combination of models and strategies. A few students, however, continued to demonstrate that they had a limited understanding of fractions. Students used versions of the clock model on each of the five items of the posttest. There were 60 instances: 41 solutions were correct and 19 were incorrect.

Section 6: Results of the Retention Test and Analysis

The retention test was written six weeks after the posttest, near the end of June. Topics of instruction during the six weeks included decimals, probability, geometry, and algebra.

Understanding fraction symbols

An analysis of all of the students' responses to the retention test confirms that all but two students were able to interpret fraction notation at the time of the retention test. These two students showed limited understanding of fractions throughout the unit.

Comparing fractions

Compare unit fractions. For the Compare question, "Which fraction is larger, $^{1}/_{6}$ or $^{1}/_{8}$?", about the same number of students answered correctly on the retention test (18 students, 82%) compared to the posttest (19 students, 86%). Students gave a variety of different explanations. The principal difference between the responses on the posttest and the retention test was the number of students who made drawings. On the retention test, only five students made drawings (two correct, three incorrect) compared to 13 (11 correct, two incorrect) on the posttest. In general, for the students who did not make a drawing, rather than having to work out a solution based on a drawing, these students had constructed the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p. 15) and used this, or a version of it, as their explanation. I believe that, at the time of the posttest, students still had a need to make a diagram to refer to but by the time of the retention test were more certain of using the big idea to justify their response. Perhaps one way to look at it is their construction of the big idea had progressed from somewhat fragile to much stronger.

Equivalence. Sixteen students (73%) answered the Equivalence question (Does $^{4}/_{6}$ = $^{2}/_{3}$?) correctly on the retention test, compared to 19 students (86%) on the posttest. Thirteen students made drawings in their responses to the Equivalence question on the retention test, compared to fourteen on the posttest. The drawings made by students on the retention test included: rectangles (six students), circles (six students), and clock (one student). The student who drew the clock model compared areas in order to work out the correct answer. The students who drew rectangles used matching areas or the unit fraction as a reference to justify their answers. Of the six students who used circle models, three responses were correct and three were incorrect. I contend that it has to do with how they are using their drawings. I think that the three students with incorrect answers relied heavily on the accuracy of their drawings because they had not drawn the sixths correctly (P436, P440, P445), as shown in Figure 5. Of the students who used a circle model to get the correct answer, one drew the sixths correctly; the other two did not. I think these two students were using the drawing as a tool for thinking (Gravemeijer, 1999) rather than as a concrete model and that they were reasoning about the size of the pieces. I think they understood the relationships and were not relying on the accuracy of their drawings (P433, P444) as shown in Figure 6. I saw this with the use of the rectangle also; one student compared rectangles that were not the same size whole and yet, reasoned for the correct answer (P427). If the student had been relying on the drawings she would have answered that the fractions were not equal since they did not look equal.

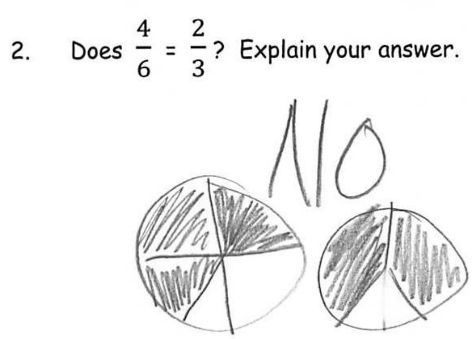

Figure 5. P440 Student reliance on inaccurate drawings

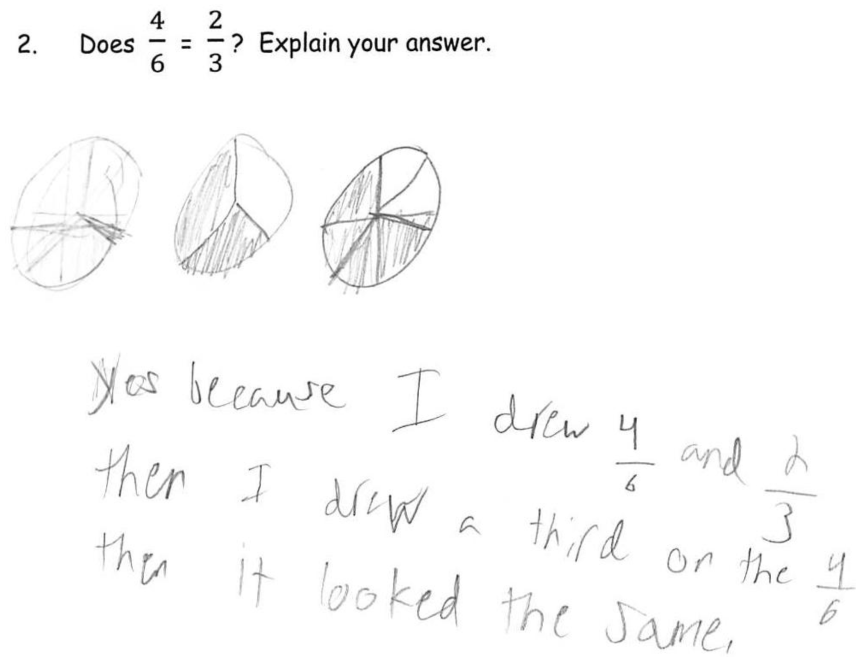

Figure 6. P433 Student use of drawing as a tool for thinking

Another aspect to consider is that there were seven students who answered the Equivalence question correctly without the use of a diagram. Three used only the unit fraction as a reference i.e. $^{2}/_{6} = {}^{1}/_{3}$ so $^{4}/_{6} = {}^{2}/_{3}$ (P431, P434, P435). These students clearly stated the relationship between sixths and thirds. The others each had different explanations, such as equivalent fractions based on the clock model or using the numbers of minutes in an hour by skip counting, or writing about overlaying fractions from an area model but not actually making a drawing. The students who did not require the use of the diagram and used the big ideas as justification for their answers had begun to generalize.

Adding fractions

Exercise. Seventeen students (77%) answered the Exercise question ($^{1}/_{6} + {}^{1}/_{2}$, in context) correctly. The most common correct solution involved students thinking of the fraction as the number of minutes and reporting their answer in number of minutes (eight students, 36%) (e.g., P478). Of these eight students, only one was able to correctly answer the bare addition questions. He used words to express equivalent fractions (e.g., P460). The others attempted unsuccessfully to use other models on the bare addition questions, although one used clock facts in minutes on both bare questions and another student used it on Add Twelfths. The other six students did not use clock facts in minutes on the bare questions. The second most common solution was similar; however, students followed through with reporting the answer as a fraction of an hour (five students, 23%) (e.g., P469). Two students used the clock as an area model and two used the open clock. Four students answered incorrectly, including two who were not able to work with the models they had chosen (one rectangle, one circle). One worked with the clock area model and made a mistake when repartitioning; one used clock facts but wrote that half an hour was 60 minutes. One student did not respond to the question.

Add sixths. Eight students (36%) were able to answer the Add Sixths question ($^2/_3 + {}^1/_6$, bare) correctly. Three students used clock equivalence facts and wrote their answer in terms of a fraction. One of these students noted that, at first, she was not able to answer it but got an idea from the Exercise question (P447). Two students used the circle model to work out the correct answer; one used equivalent fractions; one used the clock area model; and one showed no strategy, but I believe he was able to generalize and visualize rectangles in his head, based on other work he had done. He reported an answer of "$^5/_6$ or $^{2.5}/_3$" (P452). The two most common problems were that students were not able to numerically combine fractions that they were able to represent with drawings (four students: two drew circles, two drew rectangles) and various types of N-distractor errors (four students).

Add twelfths. Six students (27%) gave correct answers to the Add Twelfths question ($^3/_4 + {}^2/_{12}$, bare). Four students successfully used clock facts and represented their answers as fractions (e.g., P490). One student used a circle model and one used a rectangle model in combination with equivalent fractions. With regards to the incorrect responses (64%), three students could draw out the answer but were not able to combine the fractions and report a numerical answer. Two of these students used a circle model and one used a clock area model. Three students had N-distractor errors and did not use any diagrams. Two students used clock facts but left their answers in the form of whole minutes instead of changing back to fractions. Two students used rectangle models very poorly.

Summary

The percentage of correct responses on the retention test followed a pattern similar to the posttest, confirming that students had constructed a depth of understanding of fractions that continued over the six weeks from the end of the unit to the retention test. Two students in the class consistently demonstrated a very limited understanding of fractions which continued through to the retention test. "Variety" is the key word to describe student work: Students used a variety of models and strategies to work out solutions, and many used big ideas to justify their responses. Most of the students (73-82%) were able to correctly compare fractions while a smaller portion of the students (27-36%) were able to add bare fractions. About three quarters of the class was able to successfully add fractions in a context involving time.

Section 7: Overview and Summary of Results

It is fruitful to examine the results over time for the five items that were asked across all of the tests. For the complete question matrix of items across tests see Appendix N. The item wording and percentage of correct answers are displayed in Table 7 and Figure 7 below. The items are grouped into Compare or Addition items.

Table 7

Percentage of Correct Answers on Corresponding Questions from Pretest (N = 22), Midtest (N = 23), Posttest (N = 22), and Retention Test (N = 22)

Question Short Form	Question Wording	Pretest %	Midtest %	Posttest %	Retention Test %
COMPARE					
1 Compare unit fractions	Pretest: Put these fractions in order from smallest to largest: $^1/_5$, $^1/_3$, and $^1/_4$ (Cramer & Henry, 2002, p. 43)	23	-	-	-
	Which fraction is larger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)	-	78	86	82
2 Equivalence	Does $^4/_6 = ^2/_3$? Explain your answer. (adapted from Van de Walle et al., 2011, p. 310)	-	65	86	73
ADDITION					
4 Exercise	Joel exercised at the Complex one day last week. He ran really fast for $^1/_6$ of an hour and then ran at his normal pace for $^1/_2$ an hour. How much of the hour did he spend running altogether? Explain your answer. (adapted from Imm et al., 2007, p. 24)	-	17	68	77
3 Add sixths	Add. $^2/_3 + ^1/_6$	-	4	41	36
5 Add twelfths	Add. $^3/_4 + ^2/_{12}$ (Imm et al., 2007, p. 24)	14	-	36	27

Note. – not asked

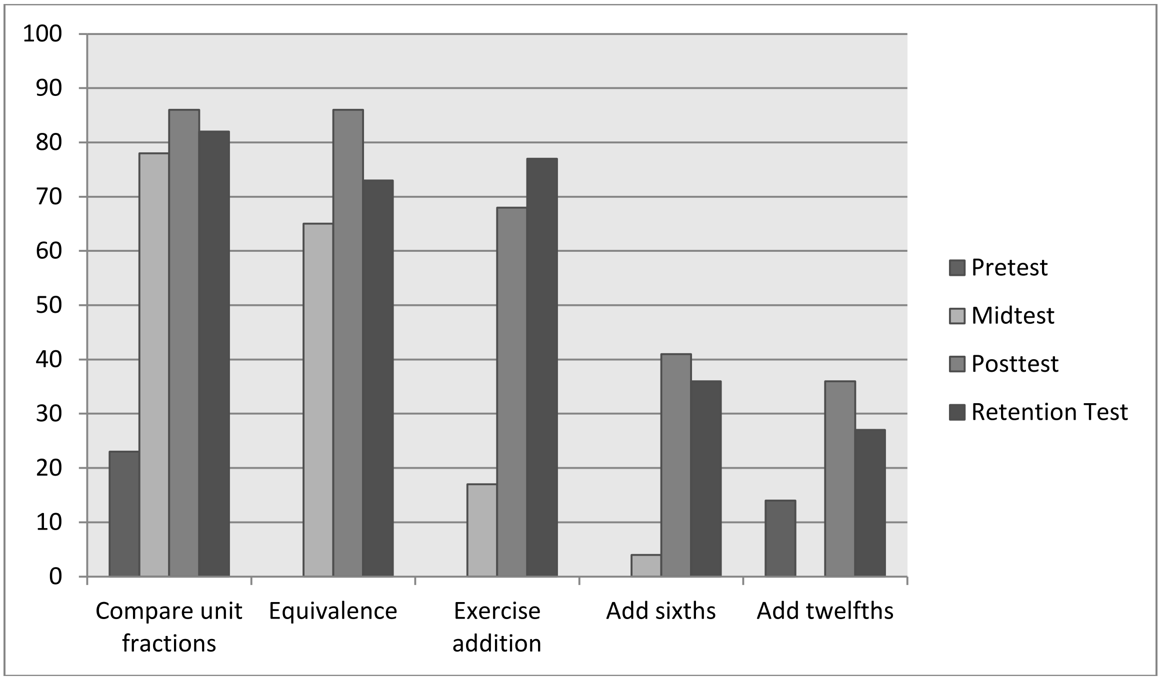

Figure 7. Percentage of correct answers on pretest ($N = 22$), midtest ($N = 23$), posttest ($N = 22$), and retention test ($N = 22$). Questions 2, 3, and 4 were not asked on the pretest. Question 5 was not asked on the midtest.

Comparing fractions

The instruction in first part of the fractions teaching unit helped most students to begin to construct the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p. 15) in order to compare unit fractions. The percentage of students who answered this question correctly on subsequent tests increased slightly as their understanding of the idea became stronger.

In general, by the time of the midtest, the majority of students had developed a strategy to determine the equivalence of fractions. Some students used big ideas as justification for fraction equivalence.

Adding fractions

Only a few students were able to solve the Exercise Addition item before the intervention. The percentage of correct answers increased dramatically after the co-construction and use of the clock model in class, and even increased six weeks later on the retention test.

Students had more difficulty with the bare addition questions, yet a portion of the students (27-36%) were able to develop a strategy or model to solve these problems.

Section 8: Further Analysis of Student Use of the Clock Model

Math educators have suggested teachers look upon the clock model as a way to support students' understanding of and facility with fractional computation. Cameron et al. (2006), for example, authored a facilitator's guide designed to be used to lead teacher workshops in the use and development of strings. Some of the strings for fractional calculations focused on the clock model. They suggest that "there are two ways to model strategies with the clock. One is a rotational model (where the fraction is represented by how far the hand has moved around the clock face); the other is an area model (where the fraction is represented as a portion of the clock)" (p. 29).

In my classroom research I found that the clock model was used by the students in fact in four different ways: clock area (as expected), clock facts – fractions (clock equivalent fractions), clock facts – minutes (whole numbers), and the open clock. Students in the class did not make use of the rotational model described by Cameron et al. A detailed description of each of the four ways students used the clock model is given below as well as samples of student work (see Figures 8 – 11).

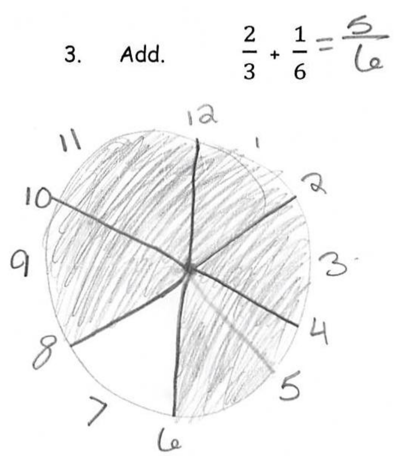

Figure 8. P342 Clock area model

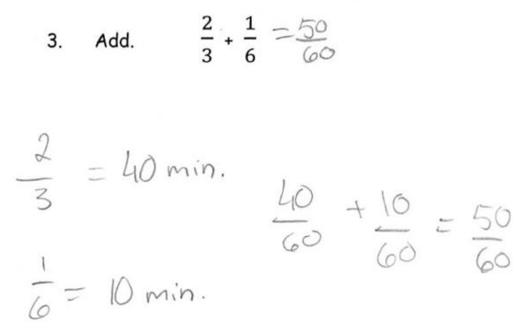

Figure 9. P348 Clock facts – fractions

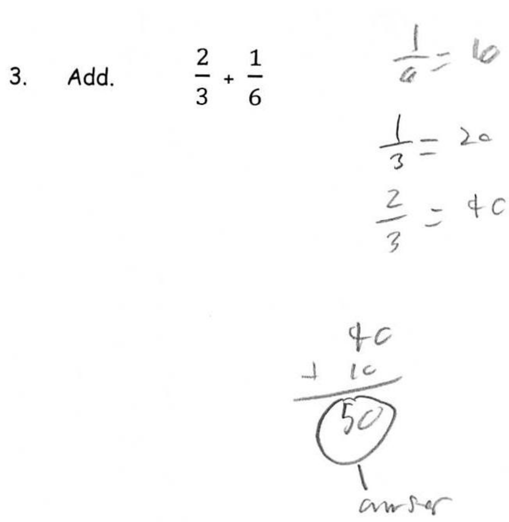

Figure 10. P351 Clock facts – minutes

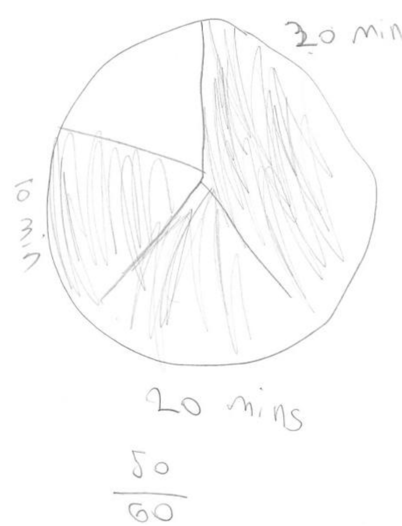

Figure 11. P346 Open clock

Clock area

Use of the clock as an area model means that a circle is drawn and the numbers 1 to 12 are written on the clock to designate each hour. Lines are drawn and sectors are shaded to represent the fractions. Seven students used the clock area model to represent their solutions.

The clock area model was used primarily on the post- and retention tests, after co-construction, with 23 occurrences. About half of these were incorrect (12 correct, 11 incorrect) due to three main reasons. First, one student was able to draw the fractions on the clock but was not able to combine them together numerically. She was not able to create equivalent fractions by subdividing the areas as other students had done. The second problem was that two students converted the shaded fractions into whole numbers of minutes for bare questions. A third reason for the high proportion of incorrect answers was that students who had a weak understanding of fractions attempted, unsuccessfully, to use this model.

One student had great success using this model not only for adding in the context of time (P364), but also for adding bare questions (see Figure 8, P342) and examining fraction equivalence (P512). The first student to use the clock area model, on the pretest Exercise Landmarks item (P177), met with limited success later, using it once in context to work out a correct answer (P467) and twice on bare addition questions, but not quite getting the correct answer (P336 and P381). One of the two students who demonstrated limited understanding throughout the duration of the project attempted, unsuccessfully, to use this model.

Using the clock area model is similar to using a simple circle area model, but it has the distinct advantage of the markings of the hourly numbers. The numbers on the clock face can be very useful to students when dividing the clock into sections to represent fractions. Compare the solutions given by two students to the Add Sixths question on the retention test (P513 and P457). In the first example, Figure 12, the student used the clock area model. The numbers on the clock facilitated the accurate drawing of the fractions. In the second, the drawings were inaccurate: the thirds are out of proportion, as shown in Figure 13. I have observed that it is difficult to accurately draw thirds and sixths on a circle. Thus there is an advantage of using the clock area model over a circle area model for fractions that are easily represented on a clock such as thirds, sixths, and twelfths.

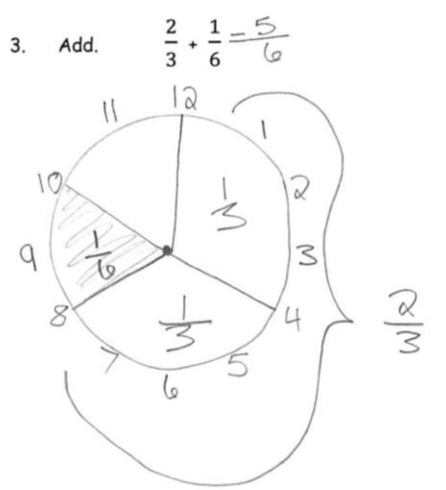

Figure 12. P513 Clock area model correctly used for Add Sixths on retention test

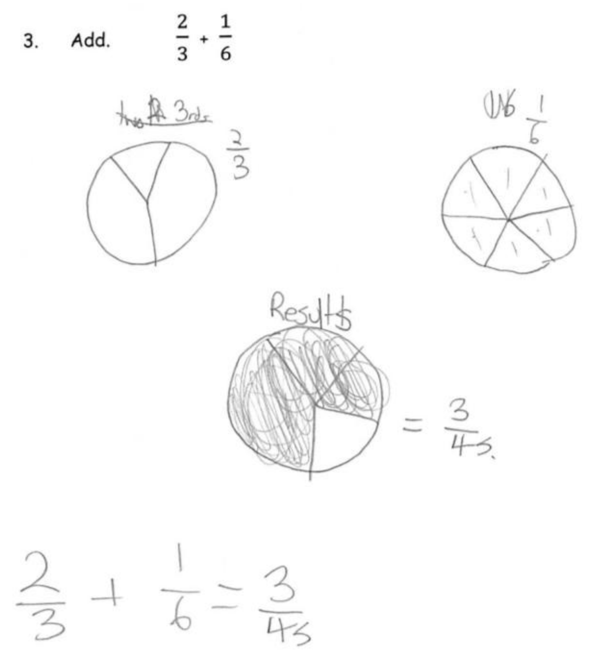

Figure 13. P457 Circle model used incorrectly for Add Sixths on retention test

Clock facts – fractions

The use of the clock model equivalent fractions means that a fraction is expressed in terms of a portion of an hour: the number of minutes of an hour with the denominator 60. Sometimes students wrote their answer in terms of twelfths, the number of 5-minute chunks or even sixths, the number 10-minute chunks. The representation of fractions as clock facts with equivalent fractions is a model that was used successfully 22 times on the post- and retention tests. In all, nine students used the clock model in this fashion. It was used unsuccessfully only once, on the midtest where the student thought $^{1}/_{6}$ of an hour was 15 minutes or $^{15}/_{60}$. The student did not go on to use the equivalent fractions representation on later tests. Eight students used clock facts – fractions successfully for addition of context and bare items. One student even used it to examine fraction equivalence on the posttest (P326).

Clock facts – minutes

The use of the clock facts – minutes means that a fraction is converted into whole numbers of minutes, similar to using a double number line, but not converted back into a fraction. This representation was used 33 times on the post- and retention tests, with 20 correct and 13 incorrect. This representation is satisfactory for answering a context question in terms of minutes. The problem occurs when students use it to answer a bare question and leave the answer in terms of whole numbers, as shown in Figure 10 (P351). Students who are more comfortable working with whole numbers of minutes than with fractions, especially students who do not fully understand the relationship between the numerator and the denominator, do not realize that a whole number answer for a bare question does not make sense (e.g., $^2/_3 + {}^1/_6 = 50$). Sometimes students will compensate for their lack of understanding of fractions in a similar way by consistently expressing fractions as decimal equivalents in order to work with only one number. The other common error made by students was a mistake in either recalling or figuring out the equivalence, for example, writing that $^1/_6$ was 15 minutes. Sixteen students used this representation at least once over the four tests.

Open clock

The open clock model is a circle drawn to represent a clock but the numbers for the hours are not written on it. The clock is "open" in the same sense as a number line with no markings on it is an open number line. For example, when solving the Add Sixths question, students marked $^1/_2$ and $^1/_6$ of the circle and sometimes labeled them with 30 minutes and 10 minutes; other times that information was recorded beside or below the drawing, or not at all. Students either knew, or could quickly figure out, how many minutes were in each sector. Students did not label each hour on the clock, as was the case for the clock area model. Eight students used the open clock model. The open clock was used 17 times, with 15 correct and two incorrect. Fifteen uses were on the posttest and only two on the retention test. The incorrect answers were both given by the same student, who added unit fractions instead of common fractions, for example, added $^1/_3 + ^1/_6$ instead of $^2/_3 + ^1/_6$. Seven, or about half, of the uses of the open clock were on the posttest Exercise Addition question, $^1/_6 + ^1/_2$. The open clock was used by students only on addition questions, not for the equivalence or compare questions.

Summary

Children learned to use clock facts to help them work with fractions, for example, $^1/_2$ hour is $^{30}/_{60}$ or 30 minutes. The weakest use of the clock model was clock facts as whole numbers of minutes. Students who had difficulty with fractions tended to make the error of not converting their answer back into fractional form. Clock facts with equivalent fractions proved to be a useful model, providing a common whole and facilitating the comparison and addition of fractions with unlike denominators. The clock area model was used as a starting point for students who could represent fractions on the face of a clock, making use of the hourly markings. The open clock was used by students as a tool to record or visualize their thinking about fractions.

Section 9: Interpretation of Students' Understanding on the Mid and Posttests

I looked at the change in student understanding from midtest to posttest to determine the effect of the clock model on their understanding of fractions. I created charts and placed students' names on the charts based on the understanding they demonstrated on the midtest and posttest. To do this, I looked carefully at each strategy and model they used to solve the problems. I clustered the results of the students together at the midtest (the beginning of the clock unit) into three groups: students who were performing below (not able to compare and order fractions or demonstrate and explain the relationship between equivalent fractions), at (able to compare and order fractions and demonstrate and explain the relationship between equivalent fractions), or above grade level (able to compare and order fractions, demonstrate and explain the relationship between equivalent fractions, and add fractions). I looked at the posttest to determine improvement and report on no improvement, some improvement and substantial improvement for students in each cluster. The following paragraphs outline the results of this analysis.

Students below grade level at the midtest

There were nine students who performed below grade level on the midtest. These students were not able to compare and order fractions or demonstrate and explain the relationship between equivalent fractions on the midtest. They can be grouped as follows: three who had no change from the midtest to the posttest, two who showed some improvement, and four who displayed substantial improvement. The three who had no change from the midtest to the posttest made some attempts to use a version of the clock model on the posttest but were not successful. In addition, they were not able to expand their use of other models and strategies. The two students who improved slightly from the midtest to the posttest were successful using the open clock model on the Exercise question on the posttest. The four students who showed substantial improvement moved from performing below grade level on the midtest to performing above grade level on the posttest. They used the open clock, clock facts in fractions, or clock facts in minutes to answer questions on the posttest. Three of the students were more successful than the fourth because the fourth student used clock facts in whole numbers of minutes, even when the questions required a fractional answer. The students who improved the most had adopted a new model, the clock model, and were able to recognize fractions that could be modeled with the clock and used clock facts in fractions or the open clock to solve problems.

Students at grade level at the midtest

There were nine students who performed at grade level on the midtest. They were able to compare and order fractions and demonstrate and explain the relationship between equivalent fractions. Of these nine students, three did not improve, three showed some improvement, and three demonstrated a large improvement. Of the three who did not improve, two did not adopt the clock model at all; one attempted unsuccessfully to use clock facts in minutes on the Exercise question on the posttest. The three students who showed some improvement were able to answer the Exercise question on the posttest, which they had not been able to do on the midtest. The models they worked with were the clock area model and the open clock. A further three students were able to answer the bare questions on the posttest as well as the Exercise question. Of these three, two used clock facts - fractions, clock facts - minutes and the open clock. The third student was quite successful using rectangles, and he used equivalent fractions in sixtieths to answer the Exercise question on the posttest. He did not shift to using the clock model but was able to use the rectangular model as a tool for thinking. Also, he had begun to generalize, citing unit fraction equivalence as a reference without any drawings. The students who showed no improvement did not use the clock model, or had attempted unsuccessfully to use it, and were not able to modify the models they were already working with, namely rectangles and circles, to solve the problems. The students who improved made use of the clock model, or, in the case of one student, were able to adapt the rectangle model to solve the problems.

Students above grade level at the midtest

There were four students who performed above grade level expectations on the midtest. These students were able to compare and order fractions and demonstrate and explain the relationship between equivalent fractions and answer the Exercise question on the midtest. Out of these four students, three improved slightly and for one, there was no way to ascertain improvement, as she had already answered all of the questions on the midtest correctly, including the bare addition question. The three who improved slightly were able to adopt a version of the clock model on the posttest to answer all three addition questions. One used the clock area model exclusively; a second used clock fractions exclusively; and the third used the open clock exclusively. The fourth student had answered all of the questions successfully on the midtest using a rectangular model. On the posttest she switched to a circular model, with the exception of using the open clock to answer the Exercise question; otherwise she did not use the clock model. These students had used rectangles and circles with success on the midtest, but incorporated the use of the clock model where it suited the context or the numbers in the question.

Summary

Students who showed no improvement from the midtest to the posttest did not adopt the clock model and were not able to generalize other models (i.e. rectangle and circle) to deal with the fractions in the problems. Students who made some improvement solved the Exercise context problem, which most had not been able to solve on the midtest. The class time spent on discussions about the clock would have contributed positively to this development, probably even if the clock model had not been co-constructed. Students who demonstrated a substantial amount of improvement used versions of the clock model to successfully solve the bare addition problems in addition to the Exercise problem. I believe they were able to look at the numbers and decide that it would be a better choice than the rectangle model, since the kit they had built included halves, fourths, eighths and sixteenths, rather than thirds, sixths and twelfths. Two students who were very successful without using the clock model were able to use the rectangle and/or circle models effectively from early in the unit.

Section 10: Interpretation of Student Use of Models

I looked specifically at the students' use of models on the mid-, post-, and retention tests in order to make comparisons regarding the number of uses and success rate with three different models. The results of this analysis are shown in Table 8.

Table 8

Comparison of Students' Use of Models on Mid-, Post-, and Retention Tests

Question	Number of successful uses of model/number of total uses of model								
	Midtest			Posttest			Retention test		
	Clock	Circle	Rectangle	Clock	Circle	Rectangle	Clock	Circle	Rectangle
Add Sixths	0/0	0/3	1/8	7/13	1/3	1/2	4/5	2/6	0/4
Exercise	1/4	1/5	1/8	14/17	0/0	0/1	17/17	0/0	0/0
Add Twelfths				7/13	0/3	1/4	4/8	1/4	1/3

Note. Add Twelfths item was not asked on the midtest.

Models used

Two main models for fractions were co-constructed with the class in different ways, the first being the rectangular fraction kit, a physical model, and the second being the clock model. The circle model was also used by some students.

Rectangle. During the fractions teaching unit, the teacher led the students in the building of their own rectangular fraction kits. While they were cutting the pieces they were thinking and talking about the fractions represented by the pieces, their meanings and relationships among them. The kits were available to the students for use at any time during the unit. The students became familiar with the pieces by using them to play games and solve problems. The fraction kit pieces comprised halves, fourths, eighths, and sixteenths. Students were not able to directly use their kits to solve the problems on the mid-, post-, and retention tests because the fractions on these tests included thirds, sixths and twelfths in addition to halves, fourths and eighths. A few students were able to successfully generalize and draw rectangle models with thirds, sixths or twelfths to solve some or all of the problems on these tests. Many other students looked at the numbers in the problems and chose to use a version of the clock model to solve them.

Clock. Only a few students made reference to the clock on the midtest, even for the Exercise problem. The clock model was co-constructed during the second half of the teaching unit. The teacher prepared a string of questions designed to evoke the use of the clock model and posed the questions within the context of her running program. Students participated in the discussion and shared their thinking. On the days following, students had the opportunity to solve fractions problems for which the clock model could potentially have been used. Students used the clock model to demonstrate and explain the relationship between equivalent fractions. Students also used the clock model to add fractions by expressing each addend as an equivalent fraction in terms of minutes (sixtieths), five-minute chunks (twelfths), and ten-minute chunks (sixths).

Circle. The circle model was used in class with the sharing cookies activity during the third lesson. It tends to be used often by students, perhaps because of the many food items which are round and can be cut into pieces. In addition, the part-whole circle area model is often over emphasized in textbooks. In fact, it can be quite difficult to partition a circle accurately into pieces that are not a direct result of the repeated halving process, so the use of the circle model was not emphasized in class. Despite the fact that its use was not a focus, some children continued to use it, often with poor results. It may be that students continued to use the circle model as a result of instruction received in previous grades.

Possible factors affecting adoption of the clock model

Every student made an attempt to use a version of the clock model at least once. Some students did not have any success using it and others were very successful.

Success with other models. Some students made good use of the rectangle or circle models early in the unit. They were successful at solving problems with these models and so did not have a need to adopt the clock model. They were able to generalize the use of the rectangle and circle models for use with thirds, sixths, and twelfths on bare problems.

Limited understanding. Other students did not adopt the clock model for different reasons. Some students did not initially make good use of the rectangle or circle models. They were not able to generalize those models for use with thirds, sixths, and twelfths. In retrospect, I wish I had included a test question that could have been solved by using the kit in order to see if students could have used the basic rectangle model. I think the clock model was too advanced for these students.

Familiarity with the analogue clock. The fractions in the Exercise problem on the pretest were chosen because $^1/_2$ an hour and $^1/_4$ of an hour are landmark fractions that I expected students to recognize from daily life. The results from the pretest showed that only 27% of the class (six students) was able to solve the problem. It may be that students made use of digital clocks on a regular basis, rather than analogue clocks, so they were not as familiar with the equivalence in minutes as expected for this grade level. In Grade 3 in Ontario, students are expected to be able to read time to the nearest five minutes using analogue clocks (MOE, 2005). It may be that instruction they received in prior grades was not sufficient for them to work with the minutes on the clock without further instruction. Indeed, 77% of the class (17 students) answered the Exercise question correctly on the retention test, all using the clock model. This version of the Exercise problem included the fractions $^1/_6$ and $^1/_2$. Notably, $^1/_6$ of an hour is not in common usage in daily life and yet students were successful in solving the problem. Thus the class time spent co-constructing the clock model also helped students make gains in the Measurement strand of the curriculum.

Development of number sense. Students who adopted the clock model and successfully solved problems with it were able to look at the numbers in the question and decide that the clock model would be useful. They were able to create and work with equivalent fractions based on the clock model. Use of the clock model helped some students gain a deeper understanding of equivalent fractions. For example, on the Equivalence question on the midtest, only two students referenced the unit fraction (i.e. $^2/_6 = {}^1/_3$ so $^4/_6 = {}^2/_3$) while on the posttest five additional students used this relationship to explain the equivalence of the fractions.

CHAPTER FIVE

Conclusion

Summary of the Major Findings

This study was conducted to determine the impact of the co-construction and use of the clock model on students' understanding of fractions. The clock model was one of two models co-constructed during the unit and it was used by students in a number of forms. Some students did not adopt the clock model while others used it with great success. Co-construction and use of the clock model helped most students to move beyond using only landmark fractions of an hour to being able to work with thirds and sixths of an hour. The clock model helped some students to better understand the relationships between equivalent fractions.

The initial two-week period of fraction instruction was designed to address common areas of difficulty for students. Sixty-one percent of the class performed at or above grade level expectations on the midtest. These students demonstrated that they had constructed the big idea "with unit fractions, the greater the denominator, the smaller the piece is" (Fosnot, 2007, p. 15) and that they had developed strategies and/or models to explain fraction equivalence. At the time of the retention test, 73% of the class performed at or above grade level, an increase of 12% after many weeks. There are at least two factors which may have caused the increase in performance. One factor is simply the extra time spent learning fractions and the other is the co-construction and use of the clock model. Taking the comparisons displayed in Table 8 into consideration, for the Add Sixths question, students were more successful using the clock model than other models to solve this problem. This is evidence that speaks in favour of investing time to co-construct the clock model with students.

For the students who did not adopt the clock model and were not able to generalize the rectangle or circle model to deal with thirds, sixths or twelfths, the instruction carried out in the teaching unit was not sufficient. I believe these students required more time and more experiences with fair-sharing problems posed in contexts and with materials that would allow them to act out the problem. The context of time, and the clock model, was probably too abstract for them to grasp.

There are two additional items that, in retrospect, would have been useful to have on the instruments. The first is a problem students could potentially solve using use their rectangular fraction kit made with halves, fourths, eighths, and sixteenths. This would give more information about the understanding of students who were not able to use the clock model or generalize the use of the kit to thirds, sixths, and twelfths. The second is a context question that students could potentially solve using the clock model, but asked in a context other than time. For example, "Alan ate $^1/_6$ of an orange. Billy ate $^1/_2$ of an orange. How much of an orange did they eat? How much was left?" (adapted from Burns, 2003, p. 102). Asking a context question that was not in the context of time would give information about the understanding of the students who had the tendency to leave their answers in whole numbers of minutes. If they solved this question using the clock model, they would, I hope, realise that 40 minutes or simply 40 does not make sense in the context.

Fosnot and colleagues (Cameron et al., 2006; Fosnot & Dolk, 2002; Imm et al., 2007) provided partial transcripts of classroom conversations during minilessons designed to elicit the use of the clock model. In the examples given by Fosnot and colleagues, the students were older—sixth grade—and did not seem to struggle with the number of minutes in an hour or finding the number of minutes in $^1/_3$ of an hour. They could easily discuss sixths, twelfths, and sixtieths of an hour. There are no samples of any written work from these students, but based on comparing the case study students' written work with the described conversations, some students in the case study were able to work with the clock model with almost the same level of flexibility as the older students.

The results of the case study support and agree with van Dijk, van Oers, and Terwel's assertion that "children in the upper grades of primary [elementary] school are capable of designing models in co-construction" (2003, p. 69). When models are co-constructed by students, the models are expected to "correspond to the thinking levels of the pupils who made them" (p. 59). The students in the case study used various forms of the clock model, likely corresponding to their level of thinking about fractions at the time.

Conclusions

Developing models with students is not often part of mathematics instruction in Canadian schools. This study has demonstrated that co-construction of the clock model has benefits for some students when learning fractions, an often challenging part of the curriculum. Students can use the clock model in ways that make sense to them, whether it is as an area model, equivalent fractions or as the open clock. The clock provides students with ways to visualize equivalent fractions.

Long term, the clock model is helpful to students as they construct various mathematical models in progression. The money model and clock model are lower on the *Landscape* (see Figure 1) and are models that students can co-construct based on their informal experiences. Imm et al. (2007) point out that the money and clock models "support strategies for addition and subtraction with landmark fractions" (p. 5). The goal is to support students as they move to the model of the double open number line. The progression of models begins with more comfortable, everyday experiences and moves toward the more abstract double open number line where students choose their own “friendly numbers” for the common whole.

Teachers should strongly consider co-developing the clock model with their students because it is an accessible model in a series of models that are useful for helping students construct a deep understanding of fractions. Teachers must take the various understandings of the students in their class into consideration when planning and look at the *Landscape* (see Figure 1) to decide when would be the appropriate time to co-construct the clock model with their students. Also, teachers may want to assess students' familiarity with the analogue clock before co-constructing the clock model.

Considerations for Future Research

There is still much research to be done in the area of student development and use of models in mathematics. An extension of this study would be to see what the students retained in the following year. If the students moved as a group to the next grade, how might the clock model shape their long-term understanding of fractions? Also related to this work, a future study could have the students build a second rectangular fraction kit containing thirds, sixths, and twelfths as well as the first kit, and co-construct the clock model and see which model they chose to work with and were able to work with more successfully. Another suggestion for further research would have students co-construct the money model as well as the fraction kits and the clock model, and then determine if they are able to select the model best suited to the numbers when solving problems.

Certain models or representations of fractions may suit particular contexts better than others. Studies that investigate which co-constructed models are most effective in different contexts could be a beneficial next step.

Investigating student use of models in other areas of the curriculum, such as the number line for addition and subtraction and the array for multiplication and division, would help educators gain knowledge about how students learn and make sense of mathematics.

This case study was conducted in a Grade 4/5 classroom. It would be useful to conduct research on the clock model with older students in Grades 7 or 8 who are more familiar with general math facts, fractions, and the clock, to see how the co-construction and use of the clock model impacts their understanding.

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Appendices

Appendix A: Parent Letter

(printed on letterhead)

February 2011

Dear Parent/Guardian of Potential Participant;

My name is Susan Girardin and I am working on my Master of Education degree at Lakehead University. My goal for my thesis is to investigate an area in mathematics where students have difficulty learning and to find ways to improve the teaching of this topic. The focus of my research is on learning fractions and the use of models to understand fractions concepts. The title of my study is "The role of co-construction of the clock model in the development of fractional understanding in a grade 4/5 classroom."

I will be observing mathematics lessons in your child's classroom during the unit on fractions. The unit will be taught for 4 weeks during February and March 2011. The students will take a pretest, midtest, posttest, and retention test (in April) to determine what they have learned in the unit. Some samples of students' work will be collected. During some of the lessons, [teacher]'s teaching methods will be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen carefully to how they have solved the problems. Their conversations may be transcribed and quoted anonymously in my final project in order to illustrate their use of models. I, [teacher], or my supervisor Dr. Lawson, may also make use of some of the edited classroom footage and work samples for professional development for teachers. Upon completion of the project, you will be welcome to obtain a summary of the research by contacting me at the phone number or email address given below, or by giving your mailing address or email address on the consent form.

Your child will not be identified in any written publication, including my master's thesis, possible journal articles or conference presentations. If video data is used for professional development, your child will be identified by first name only. The raw data that is collected will be securely stored at Lakehead University for five years and then destroyed. Participation in this study is voluntary and you may withdraw the use of your child's data at any time, for any reason, without penalty. The research project has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 343-8283 or swright@lakeheadu.ca. The research has been approved by the [school board] and the Principal of [name of] School.

Please note that this research does not affect classroom instruction time, with the exception of the 45-minute retention test. The lessons are being carried out by [teacher] in the same manner and length of time as they would be without the research project. This research will not take away from the normal learning environment in the classroom and there is no apparent risk to your child. The research is simply being conducted to make note of the effects of using a model, which is a regular part of the fractions unit. If you choose not to have your child participate, he or she will still be engaged in the math lessons. The only difference is that his or her data will not be used. If you give permission for your child to participate, your child will also be asked whether he or she is willing to take part in this research.

You are welcome to contact me at 000-0000 or username@lakeheadu.ca if you have any questions concerning this research project. I would be very pleased to speak with you.

If you agree to allow your child to participate in the study, please sign the attached letter of consent and return it to [teacher] at the school. Please keep this letter in case you would like to contact any one of us.

Sincerely,

Susan Girardin

Mrs. S. Girardin
Master's Student
Lakehead University
807-
username@lakeheadu.ca

Dr. A. Lawson, Ph.D.
Thesis Supervisor
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Principal
[Name of] School
807-

Sue Wright
Research Ethics Board
Lakehead University
807-343-8283
swright@lakeheadu.ca

Appendix B: Parent Consent Form

(printed on letterhead)

I DO give permission for my son/daughter, ____,
(Student's Name/please print)

to participate in the study with Susan Girardin as described in the attached letter.

I understand that:

1. My child will be videotaped in the classroom environment as part of the research.
2. My child's participation is entirely voluntary and I can withdraw permission at any time, for any reason, with no penalty.
3. There is no apparent danger of physical or psychological harm.
4. In accordance with Lakehead University policy, the raw data will remain confidential and securely stored at Lakehead University for five years and then destroyed.
5. All participants will remain anonymous in any publication resulting from the research project.
6. The video clips of the classroom or student work may be included in Professional Development for teachers conducted by me, [teacher], or Dr. Lawson. If my child appears in the video clips he/she will be identified only by first name.

☐ I initial this box to give permission for my child to appear in video clips which may be used for Professional Development purposes, as outlined in 6. above.

7. I can receive a summary of the project, upon request, following the completion of the project, by calling or writing, or by providing my address or email address below.

Please keep the introductory letter on file should you have any further questions.
If you agree to let your child take part in the study, please complete this page and have your child return it to [teacher].

____
Name of Parent/Guardian (please print)

____ ______________
Signature of Parent/Guardian Date

Address or email address (if you would like a summary of the findings):

____

Appendix C: Potential Participant Letter

(printed on letterhead)

February 2011

Dear Potential Participant;

In February and March I will be coming to your classroom while [teacher] teaches fractions. I will be paying attention and writing things down during your math classes because I am curious about what helps people to learn fractions best. I am a student at Lakehead University in the Master of Education program and this is part of my school project, "The role of co-construction of the clock model in the development of fractional understanding in a grade 4/5 classroom".

[Teacher] will teach the lessons to you just like she usually does. A difference you will notice is that during some of lessons there will be a video camera in the classroom and a microphone on your work table. These tools will help me with my project by recording what you say and do while you are solving problems. I, [teacher], or my supervisor Dr. Lawson, may also want to use some video clips from the classroom and samples of your work for helping other teachers learn more about how to teach about fractions. If you are in a video that will be seen by other teachers, your first name might be used. I will not use your name in anything I write about the project.

The unit will start with a pretest so that I can see what you know about fractions before any of the lessons. [Teacher] will teach the lessons and sometimes your work will be collected. I will photocopy some of the work that [teacher] collects and use it to help me understand your thinking. There will be a test in the middle of the unit to see how you are doing so far, and a test at the end to see what you have learned. Later in April there will be another test, called a retention test, to see what you remember.

Please ask me any questions you have about my project and I will be happy to answer them. You can decide whether or not to be part of my project. You will be doing the same work in math class whether you are in my project or not, the only difference is that I will not use your test results or your work or any video clips with you in them if you decide not to take part.

Thank you for thinking about being part of my project.

Sincerely,

Mrs. Susan Girardin

Appendix D: Potential Participant Consent Form

(printed on letterhead)

I, ____, want to take part in the project with
(Student's Name/please print)

Mrs. Susan Girardin as described in the letter.

I understand that:

1. I will be videotaped in the classroom as part of the project.

2. I don't have to take part in the project, but I want to be part of it, and I know I can change my mind about that later and it wouldn't be a problem.

3. It is safe to be part of this project.

4. All of the information Mrs. Girardin collects for her project will be kept in a very safe place at Lakehead University for five years and then it will be destroyed.

5. My name will never be used in anything Mrs. Girardin writes about the project.

6. Mrs. Girardin, [teacher], or Dr. Lawson might want to use some of the videos or copies of my work to help other teachers learn about teaching fractions. My first name might be used in video clips of the classroom. My name will not be on any copies of my work.

☐ I put my initials in this box to show that it is alright for me to appear in video clips which may be used for helping other teachers learn about teaching fractions.

If you want to be part of my project, please fill in this page and give it to [teacher].

____
Name of Student (please print)

____ ________________
Signature of Student Date

Appendix E: Teacher Letter

(printed on letterhead)

February 2011

Dear [teacher],

Thank you for considering participation in this study. My goal for my master's thesis is to investigate an area in mathematics where students often have difficulty learning and to find ways to improve the teaching of this topic. The focus of my research is on learning fractions and the use of models to understand fractions concepts. This study is designed to explore the impact on students' thinking when a teacher and students develop the clock model together in the context of a unit on fractions. The title of my study is "The role of co-construction of the clock model in the development of fractional understanding in a grade 4/5 classroom." Presently there is very little information available about the effects of the co-construction of the clock model.

In order to gather the information needed for the study, I will be observing mathematics lessons in your classroom during the unit on fractions. The students will take a pretest, midtest, posttest, and retention test to determine what they have learned in the unit. You will have access to the test results for student assessment. Some samples of students' work will be collected. During some of the lessons, your teaching methods will be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen carefully to how they have solved the problems. Conversations may be transcribed and quoted anonymously in my final project in order to illustrate their use of models. You, I, or my supervisor Dr. Lawson, may also make use of some of the edited classroom footage and student work samples for professional development for teachers.

As part of the project you will need to: distribute and collect the cover letters and permission forms from parents or guardians and students; collect student work; and, allow time for the tests (including the retention test). I will ensure that you have any of the resources you might need for the lessons. I hope that you will participate for the duration of the study; however, you may withdraw at any time, for any reason, without penalty, as your participation is entirely voluntary. I do not anticipate any negative consequences as a result of participation in this study.

You and your students will not be identified in any written publication, including my master's thesis, possible journal articles or conference presentations. If video data is used for professional development, your students will be identified by first name only, but if children use your surname it may be revealed. The raw data that is collected will be securely stored at Lakehead University for five years after completion of the project and then destroyed. A report of the research will be available upon request. I can be reached at 000-0000 or username@lakeheadu.ca.

The research project has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 343-8283 or swright@lakeheadu.ca.

If you agree to participate in the study, please sign the attached letter of consent and return it to me.

Sincerely,

Susan Girardin

Mrs. S. Girardin Master's Student Lakehead University 807- username@lakeheadu.ca	Dr. A. Lawson, Ph.D. Thesis Supervisor Lakehead University 807-343-8720 alawson@lakeheadu.ca
	Sue Wright Research Ethics Board Lakehead University 807-343-8283 swright@lakeheadu.ca

Appendix F: Teacher Consent Form

(printed on letterhead)

I ____, do agree to participate in the study with
(Teacher's Name/please print)

Susan Girardin as described in the attached letter.

I understand that:

1. I will be videotaped in the classroom as part of the research.

2. My participation is entirely voluntary and I can withdraw permission at any time, for any reason, without penalty.

3. There is no apparent danger of physical or psychological harm.

4. In accordance with Lakehead University policy, the raw data will remain confidential and securely stored at Lakehead University for five years and then destroyed.

5. I will remain anonymous in any publication resulting from the research project.

6. The video clips of the classroom or my work may be included in Professional Development for teachers conducted by Susan Girardin, myself, or Dr. Lawson. If I appear in the video clips I may be identified by surname.

☐ I initial this box to give permission for me to appear in video clips which may be used for Professional Development purposes, as outlined in 6. above.

____

Name of Third Party Witness (please print)

____ _______________
Signature of Third Party Witness Date

If you agree to take part in my study, please complete this page and return it to me.

____
Name of Teacher (please print)

____ _______________
Signature of Teacher Date

Appendix G: Principal Letter

(printed on letterhead)

February 2011

Dear [Principal's Name],

Thank you for considering participation in this study. My goal for my master's thesis in education is to investigate an area in mathematics where students often have difficulty learning and to find ways to improve the teaching of this topic. The focus of my research is on learning fractions and the use of models to understand fractions concepts. This study is designed to explore the impact on students' thinking when a teacher and students develop the clock model together in the context of a unit on fractions. The title of my study is "The role of co-construction of the clock model in the development of fractional understanding in a grade 4/5 classroom." Presently there is very little information available about the effects of the co-construction of the clock model.

In order to gather the information needed for the study, I will be observing mathematics lessons in [teacher]'s classroom during the unit on fractions. The students will take a pretest, midtest, posttest, and retention test to determine what they have learned in the unit. She will have access to the test results for student assessment. Some samples of students' work will be collected. During some of the lessons, [teacher]'s teaching methods will be videotaped. Also, with permission, some groups of students will be videotaped so that I will be able to listen carefully to how they have solved the problems. Conversations may be transcribed and quoted anonymously in my final project in order to illustrate their use of models. I, [teacher], or my supervisor Dr. Lawson, may also make use of some of the edited classroom footage and student work samples for professional development for teachers.

This research does not affect classroom instruction time, with the exception of the 45-minute retention test. The lessons are being carried out by [teacher] in the same manner and length of time as they would be without the research project. This research will not take away from the normal learning environment in the classroom and there is no apparent risk. If parents choose not to have a child participate, the child will still be engaged in the math lessons. The only difference is that his or her data will not be used. If parents give permission for a child to participate, the child will also be asked whether he or she is willing to take part in this research.

I hope that [teacher] and her students will participate for the duration of the study; however, you may withdraw your permission at any time, for any reason, without penalty, as participation is entirely voluntary. I do not anticipate any negative consequences as a result of participation in this study.

The School Board, [name of] School, [teacher], and her students will not be identified in any written publication, including my master's thesis, possible journal articles or conference presentations. If video data is used for professional development, the students will be identified by first name only; however, if students use the teacher's surname it may be revealed. The raw data that is collected will be securely stored at Lakehead University for five years after completion of the project. A report of the research will be available upon request. I can be reached at 000-0000 or username@lakeheadu.ca.

The research project has been approved by the Lakehead University Research Ethics Board. If you have any questions related to the ethics of the research and would like to speak to someone outside of the research team, please contact Sue Wright at the Research Ethics Board at 343-8283 or swright@lakeheadu.ca.

If you give permission for participation in the study, please sign the attached letter of consent and return it to me.

Sincerely,

Susan Girardin

Mrs. S. Girardin
Master's Student
Lakehead University
807-
username@lakeheadu.ca

Dr. A. Lawson, Ph.D.
Thesis Supervisor
Lakehead University
807-343-8720
alawson@lakeheadu.ca

Sue Wright
Research Ethics Board
Lakehead University
807-343-8283
swright@lakeheadu.ca

Appendix H: Principal Consent Form

(printed on letterhead)

I ____, do agree to participation in the study with
(Principal's Name/please print)

Susan Girardin as described in the attached letter.

I understand that:

1. [teacher] and her students will be videotaped in the classroom as part of the research.

2. Their participation is entirely voluntary and I can withdraw permission at any time, for any reason, without penalty.

3. There is no apparent danger of physical or psychological harm.

4. In accordance with Lakehead University policy, the raw data will remain confidential and securely stored at Lakehead University for five years and then destroyed.

5. The [Name of] School Board, [name of] School, [teacher], and her students will remain anonymous in any written publication resulting from the research project.

6. The video clips of the classroom or student work may be included in Professional Development for teachers conducted by Susan Girardin, [teacher], or Dr. Lawson. If students appear in the video clips, they will only be identified by first name. If [teacher] appears in the video clips, she may be identified by surname.

☐ I initial this box to give permission for [teacher] and her students to appear in video clips which may be used for Professional Development purposes, as outlined in 6. above.

If you approve of participation in my study, please complete this page and return it to me.

____
Name of Principal (please print)

____ ________________
Signature of Principal Date

Appendix I: Pretest

1. 4 children are sharing 10 brownies so that everyone gets the same amount. How much brownie can 1 person have? (Empson, 2002, pp. 29-30)

2. What does the bottom number in a fraction tell us?

   What does the top number in a fraction tell us? (Van de Walle, 2007, p. 299)

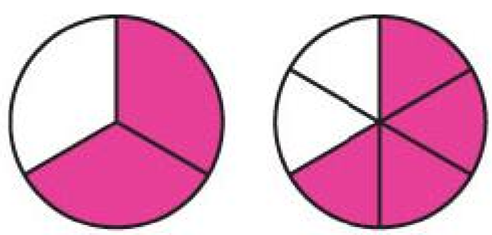
3. If Linda ate one half of an apple pie and two thirds of a cherry pie, how much did she eat? (adapted from Sharp et al., 2002, p. 21)

4. Put these fractions in order from smallest to largest: $^1/_5$, $^1/_3$, and $^1/_4$ (Cramer & Henry, 2002, p. 43)

5. Joey and Robert each had the same size pizza. Joey cut his pizza into eight equal pieces and ate six of them. Robert cut his into five equal pieces and ate four of them. Who ate more pizza? (Burns, 2001, p. 138)

6. Is the following sum larger or smaller than 1? Explain how you know. Use estimation. $^1/_2 + {}^1/_3$ (Reys et al., 1999, p. 530)

7. Are the following shaded circles equal? Compare and explain your answer. (Jigyel & Afamasaga-Fuata'i, 2007, p. 21)



8. Add. $^3/_4 + {}^2/_{12}$ (Imm et al., 2007, p. 24)

9. Joel worked out at the Complex the other day. He ran for half of an hour and then walked for $^1/_4$ of an hour. How much of the hour did he spend exercising? (adapted from Imm et al., 2007, p. 27)

Appendix J: Midtest

1. Does $^4/_6 = {}^2/_3$? How do you know?
   (adapted from Van de Walle et al., 2011, p. 310)

2. Which fraction is bigger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)

3. Add. $^2/_3 + {}^1/_6$ (Imm et al., 2007, p. 25)

4. Joel exercised at the Complex one day last week. He ran really fast for $^1/_6$ of an hour and then ran at his normal pace for $^1/_2$ an hour. How much of the hour did he spend running? (adapted from Imm et al., 2007, p. 24)

Appendix K: Posttest

1. Which fraction is bigger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)

2. Does $^4/_6 = {}^2/_3$? Explain your answer.
(adapted from Van de Walle et al., 2011, p. 310)

3. Add. $^2/_3 + {}^1/_6$ (Imm et al., 2007, p. 25)

4. Judy practiced for the track meet one day last week. She walked for $^1/_6$ of an hour and then jogged for $^1/_2$ an hour. How much of the hour dids he spend practicing altogether? Explain your answer.
(adapted from Imm et al., 2007, p. 24)

5. Add. $^3/_4 + {}^2/_{12}$ (Imm et al., 2007, p. 25)

Appendix L: Retention Test

1. Which fraction is bigger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)

2. Does $^4/_6 = {}^2/_3$? Explain your answer.
(adapted from Van de Walle et al., 2011, p. 310)

3. Add. $^2/_3 + {}^1/_6$ (Imm et al., 2007, p. 25)

4. Jason exercised at the Complex one day last week. He walked for $^1/_6$ of an hour and then ran for $^1/_2$ an hour. How much of the hour dids he spend exercising altogether? Explain your answer.
(adapted from Imm et al., 2007, p. 24)

5. Add. $^3/_4 + {}^2/_{12}$ (Imm et al., 2007, p. 25)

Appendix M: Code List

Correct
Incorrect
No response
able to draw but not numerically combine
benchmarks
common whole
decimal equivalent 0.5=1/2
did not write as a fraction
division
doubles to make equivalent fraction
drawing correct but no numerical answer given
equivalent fractions
equivalent fractions influenced by clock model
fewer pieces means pieces are larger
greater denominator smaller piece for unit fractions
grouping together
guessed
idea from another question
incomplete sharing
incorrect use of common denominators
justification incorrect
landmark fractions
Level 2
Level 3 - tool to think with
limited understanding
m: abandoned clock drawings
m: chocolate bar
m: circle - fifths drawn incorrectly
m: circle - sixths drawn incorrectly
m: circle - used poorly
m: circle - used well
m: circle model
m: circle model - one piece from each
m: clock - area
m: clock facts - fractions
m: clock facts - minutes
m: cookie
m: fair sharing
m: open clock
m: rectangle
m: rectangle - one piece from each
m: tallies represent minutes
M0: uncategorized
M1: drawing needed

M2: drawing used
M3: no drawing, generalization
match up - visualize equivalence
matching areas
mis-match between drawing and answer
missing part
more pieces means smaller pieces
N-distractor
not common whole
order of unit fractions
overlay
race order
repeated addition
represents groups and objects in groups
size of piece - one sixth is larger piece than one eighth
skip counting
slicing
smaller the denominator the larger the piece is with unit fractions
strategy not shown
transitivity
unit fraction as a reference
verbal justification
visual spatial

Appendix N: Item Matrix

Short Form	Pretest	Midtest	Posttest	Retention Test
Brownies	4 children are sharing 10 brownies so that everyone gets the same amount. How much brownie can 1 person have? (Empson, 2002, pp. 29-30)			
Meaning	What does the bottom number in a fraction tell us? What does the top number in a fraction tell us? (Van de Walle, 2007, p. 299)			
Pies	If Linda ate one half of an apple pie and two thirds of a cherry pie, how much did she eat? (adapted from Sharp et al., 2002, p. 21)			
Order	Put these fractions in order from smallest to largest: $^1/_5$, $^1/_3$, and $^1/_4$ (Cramer & Henry, 2002, p. 43)			
Pizza	Joey and Robert each had the same size pizza. Joey cut his pizza into eight equal size pieces and ate six of them. Robert cut his pizza into five			

	equal size pieces and ate four of them. Who ate more pizza? (Burns, 2001, p. 138)			
Estimate	Is the following sum larger or smaller than 1? Explain how you know. Use estimation. $^1/_2 + ^1/_3$ (Reys et al., 1999, p. 530)			
Compare	Are the following shaded circles equal? Compare and explain your answer. (model provided) (Jigyel & Afamasaga-Fuata'i, 2007, p. 21)			
Add twelfths	Add. $^3/_4 + ^2/_{12}$ (Imm et al., 2007, p. 24)		Add. $^3/_4 + ^2/_{12}$ (Imm et al., 2007, p. 24)	Add. $^3/_4 + ^2/_{12}$ (Imm et al., 2007, p. 24)
Exercise (landmarks)	Joel worked out at the Complex the other day. He ran for half of an hour and then walked for $^1/_4$ of an hour. How much of the hour did he spend exercising? (adapted from Imm et al., 2007, p. 27)			
Compare unit fractions		Which fraction is larger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)	Which fraction is larger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)	Which fraction is larger, $^1/_6$ or $^1/_8$? Explain your answer. (Mack, 1990, p. 22)
Equivalence		Does $^4/_6 = ^2/_3$?	Does $^4/_6 = ^2/_3$?	Does $^4/_6 = ^2/_3$?

	Explain your answer. (adapted from Van de Walle et al., 2011, p. 310)	Explain your answer. (adapted from Van de Walle et al., 2011, p. 310)	Explain your answer. (adapted from Van de Walle et al., 2011, p. 310)
Add sixths	Add. $^{2}/_{3} + {}^{1}/_{6}$ (Imm et al., 2007, p. 25)	Add. $^{2}/_{3} + {}^{1}/_{6}$ (Imm et al., 2007, p. 25)	Add. $^{2}/_{3} + {}^{1}/_{6}$ (Imm et al., 2007, p. 25)
Exercise	Joel exercised at the Complex one day last week. He ran really fast for $^{1}/_{6}$ of an hour and then ran at his normal pace for $^{1}/_{2}$ an hour. How much of the hour did spend running altogether? Explain your answer. (adapted from Imm et al., 2007, p. 24)	Judy practiced for the track meet one day last week. She walked for $^{1}/_{6}$ of an hour and then jogged for $^{1}/_{2}$ of an hour. How much of the hour did she spend practicing altogether? Explain your answer. (adapted from Imm et al., 2007, p. 24)	Jason exercised at the Complex one day last week. He walked for $^{1}/_{6}$ of an hour and then ran for $^{1}/_{2}$ of an hour. How much of the hour did he spend exercising altogether? Explain your answer. (adapted from Imm et al., 2007, p. 24)