The block copolymer and its related problems and mathematical models

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Contents

1 Introduction

By competing short and long-range interactions, there exists some energies have been introduced and studied in the mathematics. These energies are relevance to the diblock copolymer microphase separation model. The diblock copolymer is a linear-chain molecule with two sub-chains. These two sub-chains are covalently linked to each other. One of the sub-chains is N_A monomers of typed A and the other one is N_B monomers of types B. The problem of diblock copolymers was introduced by Ohta and Kawasaki [1] in 1986 at first based on a density-functional theory. Nowadays, this problem has rekindled the interest of mathematicians.

The following research of the diblock copolymers is about the droplet regime. Furthermore, a continuous study of the sharp interface of the diblock copolymers is addressed as a study of small volume-fraction asymptotic properties of a nonlocal isoperimetric functional with a confinement term. This functional is the sharp interface limit with a large number static nanoparticles as a confinement term and penalize the energy outside of a fixed region [2].

Then, the research in Gamow's liquid drop model [3] is a variant model with a general Riesz kernel and a long-range attractive background potential with weight Z. The background potential is a regularization for the liquid drop model. Also, it restores the existence of minimizes for arbitrary mass. This research resurfaced on diblock copolymers and gained attention in the field of mathematical research. [4] provides a general overview of this problem and [5, 6, 7, 8, 9, 10, 11, 12, 13] provide the studies in some spesific areas of this problem.

Finally, after studying the diblock copolymers and its related problems, the triblock copolymer is the following research to focus on. Nakazawa and Ohta address the theory of triblock copolymers in two dimensions. The triblock copolymer has been studied in [14] and [15]. The ABC triblock copolymer [16] is a linear-chain molecule with three sub-chains. These three sub-chains are covalently linked to each other. One of the sub-chains of type A

is connected to another one of type B monomers. Also, one of the sub-chains of type B is connected to another subchain of types C monomers.

This thesis paper is organized as follows. Section 2 analyzes the problem of small volume fraction limit of the diblock copolymers based on [17] and [18]. This problem is studied by two parts: the sharp interface and the diffuse interface. Section 3 analyzes the nonlocal isoperimetric problem of the droplet phase based on [2]. Section 4 analyzes a variant of the Gamow's liquid drop model with background potential based on [19]. Section 5 analyzes the energy functional of the triblock copolyerms based on [16].

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2 The Diblock Copolymers

2.1 Introduction

Ohta and Kawasaki [1] first propose a density-functional theory and develop it to a nonlocal Cahn-Hilliard free energy by a long-range interaction term. This long-rang interaction term has a connection with the sub-chains in the diblock copolymer macromolecule as

$$\frac{\varepsilon^2}{2}\int_{\mathbb{T}^n}|\nabla u|^2dx+\int_{\mathbb{T}^n}u^2(1-u^2)dx+\frac{\sigma}{2}||u-M||^2_{H^{-1}(\mathbb{T}^n)}. \tag{2.1.1}$$

This energy can be minimized by a mass or volume constraint

$$\fint_{\mathbb{T}^n} u = M.$$

In the above equation, u represents the relative monomer density. When $u = 0$, it represents the pure-A region. When $u = 1$, it represents the pure-B region. As a result, M represents the relative abundance of the A-part, which is the volume fraction of the region A.

The fine scale structure is depended on ε, σ and M by a mass constraint, and these three can not be vanished. Therefore, by choosing the $\sigma = \varepsilon\gamma$, a rescaled (2.1.1) is given by

$$\mathcal{E}^{\varepsilon}(u) := \varepsilon\int_{\mathbb{T}^n}|\nabla u|^2dx+\frac{1}{\varepsilon}\int_{\mathbb{T}^n}u^2(1-u^2)dx+\gamma||u-\fint u||^2_{H^{-1}(\mathbb{T}^n)} \tag{2.1.2}$$

The functional $\mathcal{E}$ is used to model self-assembly of diblock copolymers [20, 1]. The first term is a penalized large gradients to balance the second term. It separates the two phases smoothly. The third term - the nonlocal term is

$$||u-\fint u||^2_{H^{-1}(\mathbb{T}^n)} = \int_{\mathbb{T}^n}|\nabla w|^2dx, -\Delta w = u-\fint_{\mathbb{T}^n} u,$$

which favors the rapid oscillation. In addition, its sharp-interface limit in the sense of Γ-

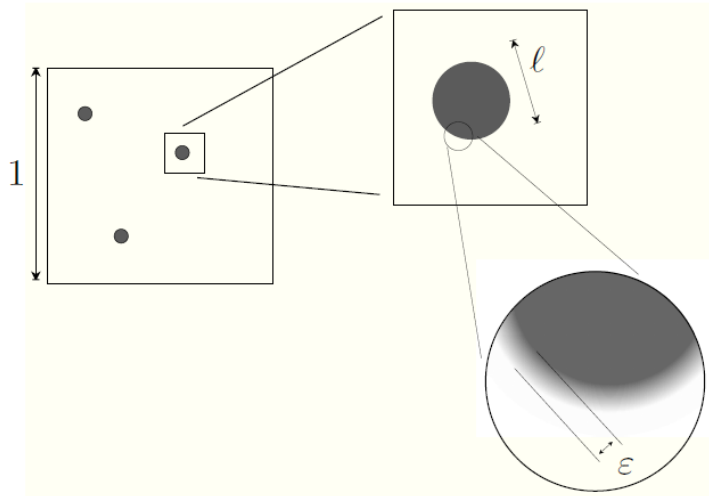

Figure 1: A two-dimensional cartoon of small-particle structures

convergence is given by [21]

$$\mathcal{E}(u) := \varepsilon \int_{\mathbb{T}^n} |\nabla u| + \gamma || u - \fint u ||^2_{H^{-1}(\mathbb{T}^n)}.$$

The most crucial part in a regime of small volume fraction is that such small regions called particles. The particles exist when $\varepsilon\sqrt{\sigma}$ is small and M is close to zero or one and both situations combined together since these two parameters control the phase diagram. The research on this topic gives the description of the energy when the volume fraction tends to zero but the number of particles in a minimizer remains $O(1)$. To achieve the goal, the limit off minimizers converge to weighted Dirac delta point measures should be examined [17]. Then, effective energetic descriptions for their positioning and local structure is able to be found. The small particle structures of the diffuse-interface are in Figure 1. The size of the periodic box $\mathbb{T}^n$ is 1. The interfacial thickness is $O(\varepsilon)$. The size of the droplets l is not fixed, which is depended on the parameter γ in $\mathcal{E}$ and volume fraction.

2.2 Definitions

Before the description of the main results, some definitions are needed to introduce. $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denotes the n-dimensional flat torus of unit volume where $\mathbb{T}^n$ is an additive group with neutral element $0 \in \mathbb{T}^n$. v denotes the characteristic function of some set A and for $v \in BV$, it is denoted as $\int_{\mathbb{T}^n} |\nabla v|$ where BV is $\mathbb{T}^n; \{0,1\}$. X denotes the space of Radon measures on $\mathbb{T}^n$. Furthermore, $\mu_\eta, \mu \in X$ and $\mu_\eta \to \mu$ denotes the weak-measure convergence.

In addition, Green's function is necessary to be introduced as $G_{\mathbb{T}^n}$ for $-\Delta$ in dimension n on $\mathbb{T}^n$ is given by

$$-\Delta G_{\mathbb{T}^n} = \delta - 1, \text{with} \int_{\mathbb{T}^n} G_{\mathbb{T}^n} = 0,$$

where the δ is the Dirac delta function at the origin. In two dimensions, for all $x = (x_1, x_2) \in \mathbb{R}^2$ with $\max\{|x_1|, |x_2|\} \leq 1/2$, the Green's function is given by

$$G_{\mathbb{T}^2}(x) = -\frac{1}{2\pi} \log |x| + g^{(2)}(x).$$

In three dimensions, for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with $\max\{|x_1|, |x_2|, |x_3|\} \leq 1/2$, the Green's function is given by

$$G_{\mathbb{T}^3}(x) = \frac{1}{4\pi|x|} + g^{(3)}(x).$$

To solve $-\Delta v = \mu$ on $\mathbb{T}^n$ for $\mu \in X$ such that $\mu(\mathbb{T}^n) = 0$, the equation is given by

$$||\mu||^2_{H^{-1}(\mathbb{T}^n)} := \int_{\mathbb{T}^n} |\nabla v|^2 dx$$

if $v \in H^1(\mathbb{T}^n)$, and then $\mu \in H^{-1}(\mathbb{T}^n)$. Furthermore, if $\mu \in L^2(\mathbb{T}^n)$ and $(u - \fint u) \in H^-1(\mathbb{T}^n)$, then the norm

$$||u - \fint u||^2_{H^-1(\mathbb{T}^n)} = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} u(x)u(y)G_{\mathbb{T}^n}(x-y)dxdy.$$

Lastly, let f be the characteristic function of a set of finite perimeter on all $\mathbb{R}^3$, $-\Delta v = f$ on $\mathbb{R}^3$ with $|v| \to 0$ as $|x| \to \infty$, and define

$$||f||^2_{H^-1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla v|^2 dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{4\pi|x-y|} dxdy.$$

2.3 Rescalings

For the sharp interface, to rescale some fixed M, a new parameter η is needed to introduce to control the vanishing volume in order to set the total mass as $\eta^n M$ [17]. Then the rescaled equation is

$$v_\eta = \frac{u}{\eta^n}$$

This lead to functionals defined over functions $v_\eta : \mathbb{T}^n \to \{0, 1/\eta^n\}$, and

$$\int_{\mathbb{T}^n} u = \eta^n M \quad \text{while} \quad \int_{\mathbb{T}^n} v_\eta = M.$$

Besides, on $v_\eta : \mathbb{T}^n \to \{0, 1/\eta^n\}$, there exist the following collection of equations

$$v_\eta = \sum_i v_\eta^i, \quad v_\eta^i = \frac{1}{\eta^n}\chi_{A_i}, \tag{2.3.1}$$

which the A_i are disjoint and connected subset of $\mathbb{T}^n$.

Since A_i have a diameter less than 1/2, then it is possible to assume that the A_i do not intersect the boundary $\partial[-1/2, 1/2]^n$. As a result, it is possible to extend V_η^i to $\mathbb{R}^n$ by defining it to be zero for $x \in A_i$. The components v_η^i to functions $z_\eta^i : \mathbb{R}^n \to \mathbb{R}$ by the mass-conservative rescaling to map their amplitude to one as

$$z_\eta^i(x) := \eta^n v_\eta^i(\eta x)$$

First, in the case of $n = 3$, the norm can be written by using the form (2.3.1) as

$$\begin{aligned} ||v_\eta - \textstyle⨍ v_\eta||^2_{H^{-1}(\mathbb{T}^3)} &= \sum_{i=1}^{\infty} \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^i(y) G_{\mathbb{T}^3}(x-y)dxdy \\ &+ \sum_{\substack{i,j=1 \\ i\neq j}}^{\infty} \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^j(x) v_\eta^j(y) G_{\mathbb{T}^3}(x-y)dxdy. \end{aligned}$$

Therefore, when the limit $\eta \to 0$, the norm can be written as

$$||v_\eta^i - \fint v_\eta^i||^2_{H^{-1}(\mathbb{T}^3)} = \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x)v_\eta^i(y)G_{\mathbb{T}^3}(x-y)dxdy$$
$$= \eta^{-1}||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} + \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x)v_\eta^i(y)g^{(3)}(x-y)dxdy \tag{2.3.2}$$

Therefore, the H^-1 norm in v is

$$\frac{1}{\eta}\sum_i ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} = O(\frac{1}{\eta}).$$

As a result, the energy is defined as

$$\mathcal{E}(u) = \int_{\mathbb{T}^3} |\nabla u| + \gamma||u - \fint u||^2_{H^{-1}(\mathbb{T}^3)}$$
$$= \eta^2\left(\eta\int_{\mathbb{T}^3} |\nabla v| + \gamma\eta^4||v - \fint v||^2_{H^{-1}(\mathbb{T}^3)}\right). \tag{2.3.3}$$

By choosing to use $\gamma = \frac{1}{\eta^3}$ and (2.3.2), (2.3.3) can be defined as

$$\mathsf{E}_\eta^{3d}(v) := \frac{1}{\eta^2}\mathcal{E}(u) = \begin{cases} \eta\int_{\mathbb{T}^3}|\nabla v| + \eta||v-\fint v||^2_{H^{-1}(\mathbb{T}^3)} & \text{if} \quad v \in BV(\mathbb{T}^3;\{0, 1/\eta^3\}) \\ \infty & \text{otherwise.} \end{cases} \tag{2.3.4}$$

Defining the energy in the case $n = 2$ is similar as case $n = 3$. When the limit $\eta \to 0$ the norm can be written as

$$||v_\eta^i - \fint v_\eta^i||^2_{H^{-1}(\mathbb{T}^2)} = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v_\eta^i(x)v_\eta^i(y)G_{\mathbb{T}^2}(x-y)dxdy$$
$$= \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} z_\eta^i\right)^2|\log\eta| - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} z_\eta^i(x)z_\eta^i(y)\log|x-y|dxdy \tag{2.3.5}$$
$$+ \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v_\eta^i(x)v_\eta^i(y)g^{(2)}(x-y)dxdy.$$

Therefore, the H^{-1} norm in v is

$$\sum_i \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} z_\eta^i\right)^2 |\log\eta| = \frac{|\log\eta|}{2\pi}\sum_i \left(\int_{\mathbb{T}^2} v_\eta^i\right)^2. \tag{2.3.6}$$

As a result, by setting $v = \frac{u}{\eta^2}$, the energy is defined as

$$\begin{aligned}\mathcal{E}(u) &= \int_{\mathbb{T}^2} |\nabla u| + \gamma ||u - \textstyle\fint u||^2_{H^{-1}(\mathbb{T}^2)} \\ &= \eta\left(\eta\int_{\mathbb{T}^2}|\nabla v| + \gamma\eta^3||v - \textstyle\fint v||^2_{H^{-1}(\mathbb{T}^2)}\right).\end{aligned}$$

By choosing to use $\gamma = \frac{1}{|\log\eta|\eta^3}$ and (2.3.5), (2.3.6), the energy can be defined as

$$\mathsf{E}^{2d}_\eta(v) := \frac{1}{\eta}\mathcal{E}(u) = \begin{cases} \eta\int_{\mathbb{T}^2}|\nabla v| + |\log\eta|^{-1}||v - \fint v||^2_{H^{-1}(\mathbb{T}^2)} & \text{if} \quad v \in BV(\mathbb{T}^2; \{0, 1/\eta^2\}) \\ \infty & \text{otherwise.}\end{cases}$$

For the diffuse interface, the rescaling process is very similar to the sharp interface. To rescale the energy ε in (2.1.2) in three dimensions, define

$$v := \frac{u}{\eta^3}$$

for some $\eta > 0$ is needed to be defined. Therefore, the energy ε is defined in v as

$$\varepsilon\eta^6\int_{\mathbb{T}^3}|\nabla v|^2 dx + \frac{\eta^6}{\varepsilon}\int_{\mathbb{T}^3}\widetilde{W}(v)dx + \gamma\ \eta^6||v - \textstyle\fint\ v||^2_{H^{-1}(\mathbb{T}^3)}, \tag{2.3.7}$$

where

$$\widetilde{W}(v) := v^2(1-\eta^3 v)^2.$$

Besides, on $v_\eta : \mathbb{T}^n \to \{0, 1/\eta^n\}$, there exist the following collection of equations

$$v_\eta = \sum_i v_\eta^i, \quad v_\eta^i = \frac{1}{\eta^n}\chi_{A_i},$$

which the A_i are disjoint and connected subset of $\mathbb{T}^n$. Then by using the Modica-Mortola convergence theorem [22] linking the perimeter to the scaled Cahn-Hilliard terms, the number of A_i remains $O(1)$ under the assumption as

$$\eta\Big[\varepsilon\eta^3\int_{\mathbb{T}^3}|\nabla v|^2dx+\frac{\eta^3}{\varepsilon}\int_{\mathbb{T}^3}\widetilde{W}(v)dx\Big]\overset{\varepsilon\ll\eta}{\sim}\eta\int_{\mathbb{T}^3}|\nabla v|=O(1).$$

The leading order of $||v_\eta-\fint v_\eta||^2_{H^{-1}(\mathbb{T}^3)}$ is $1/\eta$ since the self-interactions such that the leading order of $||v^i_\eta-\fint v^i_\eta||^2_{H^{-1}(\mathbb{T}^3)}$ is $1/\eta$. Therefore, for balancing the third term in (2.3.7), choosing $\gamma\sim 1/\eta^3$ which is

$$\gamma=\frac{1}{\eta^3}.$$

As a result, the energy is defined as

$$\mathcal{E}(u)=\eta^2\Big\{\eta\Big[\varepsilon\eta^3\int_{\mathbb{T}^3}|\nabla v|^2dx+\frac{\eta^3}{\varepsilon}\int_{\mathbb{T}^3}\widetilde{W}(v)dx\Big]+\eta||v_\eta-\fint v_\eta||^2_{H^{-1}(\mathbb{T}^3)}\Big\}.$$

Besides, since the contents of the outer parenthese is $O(1)$ as $\eta\to 0$ with $\varepsilon\ll\eta$, then the re-normalized energy is defined as

$$E_{\varepsilon,\eta}(v)=\eta\Big[\varepsilon\eta^3\int_{\mathbb{T}^3}|\nabla v|^2dx+\frac{\eta^3}{\varepsilon}\int_{\mathbb{T}^3}\widetilde{W}(v)dx\Big]+\eta||v_\eta-\fint v_\eta||^2_{H^{-1}(\mathbb{T}^3)} \tag{2.3.8}$$

2.4 Main Results

Results in Three Dimensions

For the sharp interface, in three dimensions, the results of E^{3d}_η are defined in Γ- convergence. Therefore, the Γ-limit can be defined over countable sums of weighted Dirac delta measure $\sum_{i=1}^{\infty}m^i\delta_{x^i}$.

By defining the function [17]

$$e_0^{3d}(m) := \inf\left\{\int_{\mathbb{R}^3} |\nabla z| + ||z||^2_{H^{-1}(\mathbb{R}^3)} : z \in BV(\mathbb{R}^3;\{0,1\}), \int_{\mathbb{R}^3} z = m\right\}, \tag{2.4.1}$$

the limit functional can be defined as

$$\mathsf{E}_0^{3d}(v) := \begin{cases} \sum_{i=1}^{\infty} e_0^{3d}(m^i) & \text{if } v = \sum_{i=1}^{\infty} m^i \delta_{x^i}, \{x^i\} \text{ distinct, and } m^i \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.4.1. Therefore, the first main result is stated as following. Within the space X,

$$\mathsf{E}_\eta^{3d} \xrightarrow{\Gamma} \mathsf{E}_0^{3d} \quad \text{as} \quad \eta \to 0.$$

Besides, there exists two conditions:

- Condition 1 - the lower bound and compactness: There exists a sequence $v_\eta \to v_0$ and supp v_0 is countable such that

$$\liminf_{\eta \to 0} \mathsf{E}_\eta^{3d}(v_\eta) \geq \mathsf{E}_0^{3d}(v_0)$$

  when v_η is a sequence such that the sequence of energies $\mathsf{E}_\eta^{3d}(v_\eta)$ is bounded.

- Condition 2 - the upper bound: There exists a sequence $v_\eta \to v_0$ such that

$$\limsup_{\eta \to 0} \mathsf{E}_\eta^{3d}(v_\eta) \leq \mathsf{E}_0^{3d}(v_0)$$

  when $\mathsf{E}_0^{3d}(v_0) \leq \infty$.

Moreover, there exists only a finite number of m^i are non-zero if $\{m^i\}_{i\in\mathbb{N}}$ with $\sum_i m^i \leq \infty$ satisfies

$$\sum_{i=1}^{\infty} e_0^{3d}(m^i) = e_0^{3d} \sum_{i=1}^{\infty} m^i. \tag{2.4.2}$$

Therefore, define the set of admissible limit sequences [17]

$$\mathcal{M} := \Big\{ \{m^i\}_{i\in\mathbb{N}} : m^i \geq 0, \text{ satisfying (2.4.2), such that } e_0^{3d}(m^i) \text{ admits a minimizer for each } i \Big\}. \tag{2.4.3}$$

The global minimizer of E_0^{3d} is

$$\min\left\{ \mathsf{E}_0^{3d}(v) : \int_{\mathbb{T}^3} v = M \right\} = e_0^{3d}(M).$$

Thus, the appropriately rescaled functional as the limit of $\mathsf{E}_\eta^{3d} - e_0^{3d}$ is

$$\mathsf{F}_\eta^{3d}(v_\eta) := \eta^{-}1 \left[\mathsf{E}_\eta^{3d}(v_\eta) - e_0^{3d}\left(\int_{\mathbb{T}^3} v_\eta \right) \right].$$

Then, the limiting energy functional F_0^{3d} can be defined as

$$\mathsf{F}_0^{3d}(v) := \begin{cases} \sum_{i=1}^{\infty} g^{(3)}(0)(m^i)^2 & \\ +\sum_{i\neq j} m^i m^j G_{\mathbb{T}^3}(x^i - y^j) & \text{if } v = \sum_{i=1}^{n} m^i \delta_{x^i}, \{x^i\} \text{ distinct, and } m^i \in \mathcal{M} \\ \infty & \text{otherwise,} \end{cases}$$

Note that the main part of F_0^{3d}

$$\sum_{i\neq j} m^i m^j G_{\mathbb{T}^3}(x^i - y^j)$$

is the two-point interaction energy as known as a Coulomb interaction energy.

Theorem 2.4.2. As a result, within the space X,

$$\mathsf{F}_\eta^{3d} \xrightarrow{\Gamma} \mathsf{F}_0^{3d} \quad \text{as} \quad \eta \to 0.$$

Condition 1 and Condition 2 of Theorem 2.4.1 are still hold with the replacing of E_η^{3d} and E_0^{3d} with F_η^{3d} and F_0^{3d}.

For the diffuse interface, the small behaviour of $E_{\varepsilon,\eta}$ is the crucial part to be focused on as well as the description of this behaviour via functionals over Dirac point masses. Therefore, firstly, define the surface tension as

$$\sigma := 2\int_0^1 \sqrt{W(t)}dt. \tag{2.4.4}$$

By defining the leading order function [18]

$$e_0(m) := \inf\left\{\sigma\int_{\mathbb{R}^3}|\nabla z| + ||z||^2_{H^{-1}(\mathbb{R}^3)} : z\in BV(\mathbb{R}^3;\{0,1\}), \int_{\mathbb{R}^3} z = m\right\}, \tag{2.4.5}$$

the limit functional can be defined as

$$\mathsf{E}_0(v) := \begin{cases} \sum_{i=1}^{\infty} e_0(m^i) & \text{if } v = \sum_{i=1}^{\infty} m^i\delta_{x^i}, \{x^i\} \text{ distinct, and } m^i \geq 0 \\ \infty & \text{otherwise.}\end{cases}$$

The global minimizer of E_0 is

$$\min\left\{\mathsf{E}_0(v) : \int_{\mathbb{T}^3} v = M\right\} = e_0(M).$$

Therefore, the appropriately rescaled functional as the limit of $E_{\varepsilon,\eta} - e_0$ is

$$F_{\varepsilon,\eta}(v_\eta) := \eta^-1\left[E_{\varepsilon,\eta}(v_\eta) - e_0\left(\int_{\mathbb{T}^3} v_\eta\right)\right].$$

Then, the limiting energy functional F_0 can be defined as

$$\mathsf{F}_0(v) := \begin{cases} \sum_{i=1}^{\infty} g^{(3)}(0)(m^i)^2 & \\ +\sum_{i\neq j} m^i m^j G_{\mathbb{T}^3}(x^i - y^j) & \text{if } v = \sum_{i=1}^{n} m^i\delta_{x^i}, \{x^i\} \text{ distinct, and } m^i \in \overline{\mathcal{M}} \\ \infty & \text{otherwise,}\end{cases}$$

where $\overline{\mathcal{M}} := \Big\{\{m^i\}_{i\in\mathbb{N}} : m^i \geq 0, \sum_{i=1}^{\infty} e_0(m^i) = e_0(\sum_{i=1}^{\infty} m^i)$, and $e_0(m^i)$ admits a minimizer for each $i\Big\}$.

Theorem 2.4.3. As a result, one of the main results is stated as

$$E_{\varepsilon,\eta} \xrightarrow{\Gamma} \mathsf{E}_0 \quad \text{and} \quad F_{\varepsilon,\eta} \xrightarrow{\Gamma} \mathsf{F}_0.$$

Besides, there exists two conditions:

- Condition 1 - the lower bound and compactness: Let ε_n and η_n be sequences tending to zero for some $\zeta > 0$ and $\varepsilon_n = o(\eta_n^{4+\zeta})$. There exists a sequence $v_\eta \to v_0$ and supp v_0 is countable such that

  $$\liminf_{n\to\infty} E_{\varepsilon_n,\eta_n}(v_n) \geq \mathsf{E}_0(v_0) \tag{2.4.6}$$

  when v_η is a sequence such that the sequence of energies $E_{\varepsilon_n,\eta_n}(v_n)$ is bounded.

  Moreover, the limit v_0 is a global minimizer of E_0 if $F_{\varepsilon_n,\eta_n}(v_n)$ is bounded and $\zeta \geq 1$ such that

  $$\liminf_{n\to\infty} F_{\varepsilon_n,\eta_n}(v_n) \geq \mathsf{F}_0(v_0). \tag{2.4.7}$$

- Condition 2 - the upper bound: There exist two continuous functions

  $$C_1, C_2 : [0,\infty) \to [0,\infty) \text{ with } C_1(0) = C_2(0) = 0.$$

  Let ε_n and η_n be sequences tending to zero and $\varepsilon_n \leq C_1(\eta_n)$. Then there exists a sequence $v_\eta \to v_0$ such that

  $$\limsup_{n\to\infty} E_{\varepsilon_n,\eta_n}(v_n) \leq \mathsf{E}_0(v_0) \tag{2.4.8}$$

  when $\mathsf{E}_0(v_0) \leq \infty$.

Moreover, the limit v_0 minimized E and $\eta_n \leq C_2(\eta_n)$, then there exists

$$\limsup_{n\to\infty} F_{\varepsilon_n,\eta_n}(v_n) \leq \mathsf{F}_0(v_0). \tag{2.4.9}$$

In addition, by the sharp interface functionals, the limit functional of $E_{\varepsilon,\eta}$ with ε tend to zero for fixed η is defined as

$$\mathsf{E}_\eta := \begin{cases} \eta\sigma \int_{\mathbb{T}^3} |\nabla v| + \eta ||v - \fint v||^2_{H^{-1}(\mathbb{T}^3)} & \text{if} \quad v \in BV(\mathbb{T}^3; \{0, 1/\eta^3\}) \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.10}$$

Therefore, the appropriately rescaled functional as the limit of $\mathsf{E}_\eta - e_0$ is

$$\mathsf{F}_\eta(v) := \eta^-1 \left[\mathsf{E}_\eta(v) - e_0 \left(\int_{\mathbb{T}^3} v \right) \right].$$

Theorem 2.4.4. As a result, one of the main results is stated as

$$\mathsf{E}_\eta \xrightarrow{\Gamma} \mathsf{E}_0 \quad \text{and} \quad \mathsf{F}_\eta \xrightarrow{\Gamma} \mathsf{F}_0, \quad \text{as } \eta \to 0.$$

Besides, there exists two conditions:

- Condition 1 - the lower bound and compactness: Let η_n be a sequence tending to zero. There exists a sequence $v_\eta \to v_0$ and supp v_0 is countable such that

  $$\liminf_{n\to\infty} \mathsf{E}_{\eta_n}(v_n) \geq \mathsf{E}_0(v_0) \tag{2.4.11}$$

  when v_η is a sequence such that the sequence of energies $\mathsf{E}_{\eta_n}(v_\eta)$ is bounded.

  Moreover, the limit v_0 is a global minimizer of E_0 if $\mathsf{F}_{\eta_n}(v_n)$ is bounded and $v_0 = \sum_{i=1} m^i \delta_{x^i}$, where $m^i \in \overline{\mathcal{M}}$ such that

  $$\liminf_{n\to\infty} \mathsf{F}_{\eta_n}(v_n) \geq \mathsf{F}_0(v_0). \tag{2.4.12}$$

- Condition 2 - the upper bound: There exists a sequence $v_\eta \to v_0$ such that

$$\limsup_{n\to\infty} \mathsf{E}_{\eta_n}(v_n) \leq \mathsf{E}_0(v_0) \tag{2.4.13}$$

  when $\mathsf{E}_0(v_0) < \infty$ and $\mathsf{F}_0(v_0) < \infty$.

  Moreover, there exists a sequence $v_\eta \to v_0$ such that

$$\limsup_{n\to\infty} F_{\varepsilon_n,\eta_n}(v_n) \leq \mathsf{F}_0(v_0). \tag{2.4.14}$$

  when $\mathsf{F}_0(v_0) < \infty$.

Results in Two Dimensions

In two dimensions, there are two differences compared to three dimensions. The first difference is the leading-order limiting behaviour. The second difference is the next-order behaviour. In addition, the critical point that makes the two-dimensional case special is the scaling of the H^{-1} (2.3.6).

Therefore, for the sharp interface, a function [17] is introduced for the local problem in two dimensions as

$$\begin{aligned} e_0^{2d}(m) &:= \frac{m^2}{2\pi} + \inf\left\{ \int_{\mathbb{R}^2} |\nabla z| : z \in BV(\mathbb{R}^2;\{0,1\}), \int_{\mathbb{R}^2} z = m \right\}, \\ &= \frac{m^2}{2\pi} + 2\sqrt{\pi m}. \end{aligned}$$

The lower-semicontinuous envelope function is defined as

$$\overline{e_0^{2d}}(m) := \inf\left\{ \sum_{j=1}^{\infty} e_0^{2d}(m^j) : m^j \geq 0, \sum_{j=1}^{\infty} m^j = m \right\}$$

since the function e_0^{2d} does not satisfy the lower-semicontinuity condition. Thus, the limit functional is defined as

$$\mathsf{E}_0^{2d}(v) := \begin{cases} \sum_{i=1}^{\infty} \overline{e_0^{2d}}(m^i) & \text{if } v = \sum_{i=1}^{\infty} m^i \delta_{x^i}, \{x^i\} \text{ distinct, and } m^i \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.4.5. As a result, within the space X,

$$\mathsf{E}_\eta^{2d} \xrightarrow{\Gamma} \mathsf{E}_0^{2d} \quad \text{as} \quad \eta \to 0.$$

Condition 1 and Condition 2 of Theorem 2.4.1 are still hold with the replacing of E_η^{3d} and E_0^{3d} with E_η^{2d} and E_0^{2d}.

For the next-order behaviour, the global minimizer of E_0^{2d} is defined as

$$\min\left\{\mathsf{E}_0^{2d}(v) : \int_{\mathbb{T}^2} v = M\right\} = \overline{e_0^{2d}}(M).$$

Therefore, the appropriately rescaled functional as the limit of $\mathsf{E}_\eta^{2d} - \overline{e_0^{2d}}$ is

$$\mathsf{F}_\eta^{2d}(v) := |\log\eta| \left[\mathsf{E}_\eta^{2d}(v) - \overline{e_0^{2d}}\left(\int_{\mathbb{T}^2} v\right)\right].$$

The situation so far looks similar as the three-dimensional case that the limiting weights m^i satisfies the minimality condition and the compactness condition for boundedness of the sequence F_η^{2d}. The compactness can be simply written as

$$\overline{e_0^{2d}}(m^i) = e_0^{2d}(m^i)$$

However, in two dimensions, the minimality condition is stronger than in the three dimensions. When $\{m^i\}_{i\in\mathbb{N}}$ is a solution of the minimization problem, then

$$\min\left\{\sum_{i=1}^{\infty} e_0^{2d}(m^i) : m^i \geq 0, \sum_{i=1}^{\infty} m^i = M\right\}. \tag{2.4.15}$$

Comparing to the three-dimensional case, the functional F_η^{2d} has one additional term as the limit $\eta \to 0$ as

$$\frac{1}{-2\pi} \sum_{i=1}^{\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} z_\eta^i(x) z_\eta^i(y) \log |x-y| dx dy. \tag{2.4.16}$$

Therefore, to calculate the above equation, it is needed to assume z_η^i to be a characteristic function of a ball of mass m^i since it has only balls as solution. Then. the first term in (2.4.16) is

$$f_0(m) := \frac{m^2}{8\pi}\Big(3 - 2\log\frac{m}{\pi}\Big).$$

In addition, some notation and optimal sequences need to be defined. For the notation, when $n \in \mathbb{N}$ and $m > 0$, the sequence $n \otimes m$ is

$$(n \otimes m)^i := \begin{cases} m & 1 \le i \le n \\ 0 & n+1 \le i < \infty. \end{cases}$$

Define $\widetilde{\mathcal{M}}$ as the set of optimal sequences

$$\widetilde{\mathcal{M}} := \{n \otimes m : n \otimes m \text{ minimizes (2.4.15) for } M = nm, \text{and } \overline{e_0^{2d}}(m) = e_0^{2d}(m)\}.$$

Then, the limiting energy functional F_0^{2d} can be defined as

$$\mathsf{F}_0^{2d}(v) := \begin{cases} n\Big\{f_0(m) + m^2 g^{(2)}(0)\Big\} + & \\ \frac{m^2}{2} \sum_{\substack{i,j \ge 1 \\ i \ne j}} G_{\mathbb{T}^2}(x^i - x^j) & \text{if } v = m\sum_{i=1}^n \delta_{x^i}, \{x^i\} \text{ distinct, and } n \otimes m \in \widetilde{\mathcal{M}} \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.17}$$

Theorem 2.4.6. As a result, within the space X,

$$\mathsf{F}_\eta^{2d} \xrightarrow{\Gamma} \mathsf{F}_0^{2d} \quad \text{as} \quad \eta \to 0.$$

Condition 1 and Condition 2 of Theorem 2.4.5 are still hold with the replacing of E_η^{2d} and

E_0^{2d} with F_η^{2d} and F_0^{2d}.

For the diffuse interface, the process to have the results is very similar to the sharp interface. There is a function [18] for the local problem in two dimensions as

$$\begin{aligned} e_0^{2D}(m) &:= \frac{m^2}{4\pi} + \inf\left\{\sigma \int_{\mathbb{R}^2} |\nabla z| : z \in BV(\mathbb{R}^2; \{0,1\}), \int_{\mathbb{R}^2} z = m\right\}, \\ &= \frac{m^2}{4\pi} + 2\sigma\sqrt{\pi m}. \end{aligned} \tag{2.4.18}$$

The lower-semicontinuous envelope function is defined as

$$\overline{e_0^{2D}}(m) := \inf\left\{\sum_{j\in J} e_0^{2D}(m^j) : m^j \geq 0, \sum_{j\in J} m^j = m\right\} \tag{2.4.19}$$

since the function e_0^{2D} does not satisfy the lower-semicontinuity condition. By rescaling with $v = u/\eta^2$ and $\gamma = \frac{1}{|\log\eta|\eta^3}$, the two-dimensional function of $E_{\varepsilon,\eta}$ is

$$E_{\varepsilon,\eta}^{2D}(v) := \varepsilon\eta^3 \int |\nabla v|^2 + \frac{\eta^3}{\varepsilon}\int v^2(1-\eta^2 v)^2 + |\log|^{-1}||v - \textstyle\fint v||_{H^{-1}}^2.$$

The analogous sharp-interface limit as $\varepsilon \to 0$ is

$$\mathsf{E}_\eta^{2D}(v) := \begin{cases} \sigma\eta \int_{\mathbb{T}^2} |\nabla v| + |\log\eta|^{-1}||v - \fint v||_{H^{-1}(\mathbb{T})}^2 & \text{if} \quad v \in BV(\mathbb{T}; \{0, 1/\eta^2\}) \\ \infty & \text{otherwise.} \end{cases}$$

where the σ is defined in (2.4.4). As a result, the first-order limit is defined as

$$\mathsf{E}_0^{2D}(v) := \begin{cases} \sum_{i\in I} \overline{e_0^{2D}}(m^i) & \text{if } v = \sum_{i\in I} m^i \delta_{x^i}, I \text{ is countable, } \{x^i\} \text{ distinct, and } m^i \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

For the next-order behaviour, the global minimizer of E_0^{2D} is defined as

$$\min\left\{\mathsf{E}_0^{2D}(v) : \int_{\mathbb{T}^2} v = M\right\} = \overline{e_0^{2D}}(M).$$

Therefore, the appropriately rescaled functional as the limit of $\mathsf{E}_\eta^{2D} - \overline{e_0^{2D}}$ is

$$\mathsf{F}_{\varepsilon,\eta}^{2D}(v) := |\log\eta| \left[\mathsf{E}_{\varepsilon,\eta}^{2D}(v) - \overline{e_0^{2D}}\left(\int_{\mathbb{T}^2} v\right)\right].$$

However, in two dimensions, the minimality condition is stronger than in the three dimensions. When $\{m^i\}_{i\in\mathbb{N}}$ is a solution of the minimization problem, then

$$\min\left\{\sum_{i=1}^{\infty} e_0^{2D}(m^i) : m^i \geq 0, \sum_{i=1}^{\infty} m^i = M\right\}. \tag{2.4.20}$$

Therefore, define $\widehat{\mathcal{M}}$ as the set of optimal sequences

$$\widehat{\mathcal{M}} := \{n \otimes m : n \otimes m \text{ minimizes (2.4.20) for } M = nm, \text{and } \overline{e_0^{2D}}(m) = e_0^{2D}(m)\}.$$

Then, the limiting energy functional F_0^{2D} can be defined as

$$\mathsf{F}_0^{2D}(v) := \begin{cases} n\Big\{f_0(m) + m^2 g^{(2)}(0)\Big\} + \\ \frac{m^2}{2}\sum_{\substack{i,j\geq 1 \\ i\neq j}} G_{\mathbb{T}^2}(x^i - x^j) & \text{if } v = m\sum_{i=1}^n \delta_{x^i}, \{x^i\} \text{ distinct, and } n\otimes m \in \widehat{\mathcal{M}} \\ \infty & \text{otherwise.} \end{cases} \tag{2.4.21}$$

Theorem 2.4.7. As a result, one of the main results is stated as

$$E_{\varepsilon,\eta} \xrightarrow{\Gamma} \mathsf{E}_0 \quad \text{and} \quad F_{\varepsilon,\eta} \xrightarrow{\Gamma} \mathsf{F}_0.$$

Besides, there exists two conditions:

- Condition 1 - the lower bound and compactness: Let ε_n and η_n be sequences tending to zero such that $\varepsilon_n \eta_n^{-3-\zeta} \to 0$ for some $\zeta > 0$. There exists a sequence $v_\eta \to v_0$ and supp v_0 is countable such that

$$\liminf_{n\to\infty} E^{2D}_{\varepsilon_n,\eta_n}(v_n) \geq \mathsf{E}^{2D}_0(v_0) \tag{2.4.22}$$

when v_η is a sequence such that the sequence of energies $E^{2D}_{\varepsilon_n,\eta_n}(v_n)$ is bounded.

Moreover, the limit v_0 is a global minimizer of E^{2D}_0 if $F^{2D}_{\varepsilon_n,\eta_n}(v_n)$ is bounded such that

$$\liminf_{n\to\infty} F^{2D}_{\varepsilon_n,\eta_n}(v_n) \geq \mathsf{F}^{2D}_0(v_0). \tag{2.4.23}$$

- Condition 2 - the upper bound: Let ε_n and η_n be sequences tending to zero such that $\varepsilon_n\eta_n^{-1}|\log\eta_n| \to 0$. Then there exists a sequence $v_\eta \to v$ such that

$$\limsup_{n\to\infty} E^{2D}_{\varepsilon_n,\eta_n}(v_n) \leq \mathsf{E}^{2D}_0(v) \tag{2.4.24}$$

when $\mathsf{E}^{2D}_0(v) < \infty$.

Moreover, the limit v minimized E and if $\varepsilon_n\eta_n^{-1}|\log\eta_n|^2 \to 0$, then there exists

$$\limsup_{n\to\infty} F^{2D}_{\varepsilon_n,\eta_n}(v_n) \leq \mathsf{F}^{2D}_0(v). \tag{2.4.25}$$

2.5 Proof

Sharp Interface

Let v_η be a sequence in $BV(\mathbb{T}^3;\{0,1/\eta^3\})$ such that both $\int_{\mathbb{T}^3} v_\eta$ and $\mathsf{E}^{3d}_\eta(v_\eta)$ are uniformly bounded. Define the function $w_\eta := \eta v_\eta$ satisfy $w_\eta \to 0 \in L^1(\mathbb{T}^3)$, and $|\nabla w_\eta| = \eta|\nabla v_\eta|$ bounded in $L^1(\mathbb{T}^3)$. Besides, $w_\eta^{3/2} = v_\eta$ is bounded in $L^1(\mathbb{T}^3)$ by definition.

Lemma 2.5.1. As a result, a subsequence has $v_\eta \to v_0$ as measures. Then, v_0 can be defined as

$$v_0 := \sum_{i=1}^{\infty} m^i \delta_{x^i},\ m^i \geq 0,\ x^i \in \mathbb{T}^3 \text{ distinct}, \tag{2.5.1}$$

such that $v_\eta \to v_0$ as measures after implying the Lemma I.1 (i) from [23] with $m = p = 1, q = 2/3$.

Lemma 2.5.2. To prove the lower bound on $\mathsf{E}_\eta^{3d}(v_\eta)$ and $\mathsf{F}_\eta^{3d}(v_\eta)$, under the same conditions as previous lemma, assume for some $n \in \mathbb{N}$ without loss of generality that

$$v_\eta = \sum_{i=1}^{n} v_\eta^i$$

with $w - \liminf_{\eta \to 0} v_\eta^i \geq m_0^i \delta_{x^i}$, supp $v_\eta^i \subset B(x^i, 1/4)$, dist (supp v_η^i, supp v_η^j) > 0 for all $i \neq j$, and diam supp $v_\eta^i < 1/4$. Besides, to prove the lower bound on $\mathsf{F}_\eta^{3d}(v_\eta)$, assume $v_\eta^i \to m_0^i \delta_{x^i}$ for each i. Thus,

$$\int_{\mathbb{T}^3} |x - \xi_\eta^i|^2 v_\eta^i(x) dx \geq C^i \eta^2,\ \xi_\eta^i \in \mathbb{T}^3 \tag{2.5.2}$$

where the constant $C^i > 0$.

The details of the proof of this lemma is given in [17] Section 5.4.

Proof of Theorem 2.4.1

Lower bound

Define v_η as a sequence such that the sequences of energies $\mathsf{E}_\eta^{3d}(v_\eta)$ and masses $\int_{\mathbb{T}^3} v_\eta$ are bounded. Then, a subsequence converge to a limit v_0 of the form (2.5.1) by Lemma 2.5.1. Also, a sequence v_η such that $v_\eta = \sum_{i=1}^n v_\eta^i$ with $w - \liminf_{\eta \to 0} v_\eta^i \geq m_0^i \delta_{x^i}$, supp $v_\eta^i \subset B(x^i, 1/4)$, and dist (supp v_η^i, supp v_η^j) > 0 for all $i \neq j$ by Lemma 2.5.2. Therefore, define

$$z_\eta^i(y) := \eta^3 v_\eta^i(x^i + \eta y). \tag{2.5.3}$$

Then, there exist

$$\int_{\mathbb{T}^3} v_\eta^i = \int_{\mathbb{R}^3} z_\eta^i \quad \text{and} \quad \int_{\mathbb{T}^3} |\nabla v_\eta^i| = \int_{\mathbb{R}^3} |\nabla z_\eta^i|,$$

and

$$||v_\eta^i - \fint v_\eta^i||^2_{H^{-1}(\mathbb{T}^3)} = \eta^{-1} ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} + \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^i(y) g^{(3)}(x-y)dxdy$$

by (2.3.2). Then, define

$$m_\eta^i := \int_{\mathbb{T}^3} v_\eta^i = \int_{\mathbb{R}^3} z_\eta^i.$$

Therefore,

$$\begin{aligned}
\mathsf{E}_\eta^{3d}(v_\eta) &= \sum_{i=1}^n \mathsf{E}_\eta^{3d}(v_\eta^i) + \eta \sum_{\substack{i,j=1\\ i\neq j}}^n \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^j(y) G_{\mathbb{T}^3}(x-y)dxdy \\
&= \sum_{i=1}^n \Big[\int_{\mathbb{R}^3} |\nabla z_\eta^i| + ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} \Big] + \eta \sum_{i=1}^n \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^i(y) g^{(3)}(x-y)dxdy \\
&\qquad + \eta \sum_{\substack{i,j=1\\ i\neq j}}^n \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^j(y) G_{\mathbb{T}^3}(x-y)dxdy \\
&\geq \sum_{i=1}^n e_0^{3d}(m_\eta^i) + \eta \inf g^{(3)} \sum_{i=1}^n (m_\eta^i)^2 + \eta \inf G_{\mathbb{T}^3} \sum_{\substack{i,j=1\\ i\neq j}}^n m_\eta^i m_\eta^j.
\end{aligned} \tag{2.5.4}$$

Therefore, the continuity and monotonicity of e_0^{3d} imply that

$$\liminf_{\eta\to 0} \mathsf{E}_\eta^{3d}(v_\eta) \geq \sum_{i=1}^n e_0^{3d}\Big(\liminf_{\eta\to 0} m_\eta^i\Big) \geq \sum_{i=1}^n e_0^{3d}(m^i) \geq \mathsf{E}_0^{3d}(v_0)$$

since the last two terms of (2.5.4) vanish in the limit.

Upper Bound

Let v_0 satisfy $\mathsf{E}_0^{3d}(v_0) < \infty$. Then, the infinite sum $v_0 = \sum_{i=1}^{\infty} m^i \delta_{x^i}$ can be approximated by finite sums trivially as

$$\mathsf{E}_0^{3d}\Big(\sum_{i=1}^{n} m^i \delta_{x^i)}\Big) = \sum_{i=1}^{n} e_0^{3d}(m^i) \leq \sum_{i=1}^{\infty} e_0^{3d}(m^i) = \mathsf{E}_0^{3d}(v_0).$$

Thus,

$$v_0 = \sum_{i=1}^{n} m^i \delta_{x^i}.$$

In addition, let $\epsilon > 0$ and z^i be near-optimal in the definition of $e_0^{3d}(m^i)$ such that

$$\int_{\mathbb{R}^3} |\nabla z^i| + ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} \leq e_0^{3d}(m^i) + \frac{\epsilon}{n}. \tag{2.5.5}$$

Assume the support of z^i is bounded based on the isoperimetric inequality. Then, define

$$v_\eta^i(x) := \eta^{-3} z^i\Big(\eta^{-1}(x - x^i)\Big). \tag{2.5.6}$$

Thus,

$$\int_{\mathbb{T}^3} v_\eta^i = m^i.$$

Besides, when η is sufficiently small, $v_\eta := \sum_i v_\eta^i$ is admissible for E_0^{3d} since the diameters of the supports of the v_η^i tend to zero and the x^i are distinct. Therefore,

$$\begin{aligned}\mathsf{E}_0^{3d}(v_\eta) = \sum_{i=1}^{n} \Big[\int_{\mathbb{R}^3} |\nabla z_\eta^i| + ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)}\Big] + \eta \sum_{i=1}^{n} \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^i(y) g^{(3)}(x-y)\,dxdy \\ + \eta \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^j(y) G_{\mathbb{T}^3}(x-y)\,dxdy.\end{aligned}$$

Therefore,

$$\limsup_{\eta \to 0} \mathsf{E}_\eta^{3d}(v_\eta) \leq \mathsf{E}_0^{3d}(v_0) + \epsilon.$$

□

Proof of Theorem 2.4.2

Lower Bound

Following the definition of v_η in Lemma 2.5.2, converge to

$$v_0 = \sum_{i=1}^{n} m^i \delta_{x^i},\ m_0^i \geq 0\text{and } x^i \text{ are distinct.}$$

Then, define

$$m_\eta^i := \int_{\mathbb{T}^3} v_\eta^i = \int_{\mathbb{R}^3} z_\eta^i.$$

Thus, follow (2.5.4)

$$\begin{aligned}
\mathsf{F}_\eta^{3d}(v_\eta) &= \eta^{-1}\Big[\mathsf{E}_\eta^{3d}(v_\eta) - e_0^{3d}\Big(\int_{\mathbb{T}^3} v_\eta\Big)\Big] \\
&= \frac{1}{\eta}\sum_{i=1}^{n}\Big[\int_{\mathbb{R}^3} |\nabla z_\eta^i| + ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} - e_0^{3d}(m_\eta^i)\Big] + \frac{1}{\eta}\Big[\sum_{i=1}^{n} e_0^{3d}(m_\eta^i) - e_0^{3d}\Big(\sum_{i=1}^{n}(m_\eta^i)\Big)\Big] \\
&\quad + \eta\sum_{i=1}^{n}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x)v_\eta^i(y)g^{(3)}(x-y)dxdy + \eta\sum_{\substack{i,j=1\\ i\neq j}}^{n}\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x)v_\eta^j(y)G_{\mathbb{T}^3}(x-y)dxdy.
\end{aligned} \tag{2.5.7}$$

As a result, since the boundedness of $\mathsf{F}_\eta^{3d}(v_\eta)$, continuity of e_0^{3d}, and the first two two terms are non-negative,

$$0 \leq \sum_{i=1}^{n} e_0^{3d}(m_\eta^i) - e_0^{3d}\Big(\sum_{i=1}^{n}(m_\eta^i)\Big) = \lim_{\eta\to 0}\Big[\mathsf{E}_\eta^{3d}(v_\eta) - e_0^{3d}\Big(\int_{\mathbb{T}^3} v_\eta\Big)\Big] \leq 0.$$

Under the condition (2.5.2), the sequence z_η^i is tight since it is bounded in $BV(\mathbb{R}^3;\{0,1\})$.

Thus, a subsequence converges in $L^1(\mathbb{R}^3)$ to a limit z_0^i. Then,

$$\begin{aligned}0 &\le \int_{\mathbb{R}^3} |\nabla z_0^i| + ||z_0^i||^2_{H^{-1}(\mathbb{R}^3)} - e_0^{3d}(m^i) \\ &\le \liminf_{\eta\to 0} \Big[\int_{\mathbb{R}^3} |\nabla z_\eta^i| + ||z_\eta^i||^2_{H^{-1}(\mathbb{R}^3)} \Big] - \lim_{\eta\to 0} e_0^{3d}(m_\eta^i) = 0\end{aligned}$$

by (2.5.7) and implies z_0^i is a minimizer for $e_0^{3d}(m^i)$.

As a result,

$$\begin{aligned}\liminf_{\eta\to 0} \mathsf{F}_\eta^{3d}(v_\eta) \ge \liminf_{\eta\to 0} \Bigg(&\sum_{i=1}^n \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^i(y) g^{(3)}(x-y)\,dxdy \\ &+ \sum_{\substack{i,j=1\\ i\neq j}}^n \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} v_\eta^i(x) v_\eta^j(y) G_{\mathbb{T}^3}(x-y)\,dxdy \Bigg) \\ = g^{(3)}(0) &\sum_{i=1}^n (m^i)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^n m^i m^j G_{\mathbb{T}^3}(x^i - y^j) = \mathsf{F}_0^{3d}(v_0).\end{aligned}$$

Upper Bound

Let x^i be distinct and $\{m^i\} \in \mathcal{M}$, define

$$v_0 = \sum_{i=1}^n m^i \delta_{x^i}.$$

z^i can be chosen to achieve the minimum in the minimization problem defining $e_0^{3d}(m^i)$ by the definition of $\mathcal{M}$. Based on the isoperimetric inequality, the support of z^i is bounded. Also, by (2.5.6), the function

$$v_\eta := \sum_{i=1}^n$$

is admissible for F_η^{3d} and $v_\eta \to v_0$ for some sufficiently small η. Therefore, by the second line of (2.5.4),

$$\lim_{\eta\to 0} \mathsf{F}_\eta^{3d}(v_\eta) = \mathsf{F}_0^{3d}(v_0).$$

□

Diffuse Interface

To proof Theorem 2.4.3, Theorem 2.4.4 is required. In addition, the following lemma from [18] is required in the proof.

Lemma 2.5.3. When there is a constant $C_0(\alpha),\ \alpha > 0$, for any characteristic function χ of a subset of $\mathbb{T}^3$ and $\delta > 0$, there exists an approximation $u \in H^1(\mathbb{T}^n, [0,1])$ with

$$\int_{\mathbb{T}^3} \delta|\nabla u|^2 + \frac{1}{\delta}u^2\,(1-u^2)dx \leq (\sigma+\alpha)\int_{\mathbb{T}^3}|\nabla\chi|,$$

and

$$\int_{\mathbb{T}^3}|\chi - u|\,dx \leq C_0(\alpha)\delta\int_{\mathbb{T}^3}|\nabla\chi|.$$

Lower Bound

For the lower bound, appropriate cutoffs are used to relate the approximate diffuse-interface sequence to the sharp-interface sequence with the same limit and smaller energy difference. To prove the condition 1, define ε_n, η_n and v_n be the same sequences in the theorem. Recall that the energy in the original scaling $u_n = \eta_n^3 v_n$. Thus, in terms of u_n,

$$E_{\varepsilon_n,\eta_n}(v_n) = \frac{\varepsilon_n}{\eta_n^2}\int_{\mathbb{T}^3}|\nabla u_n|^2 dx + \frac{1}{\eta_n^2\varepsilon_n}\int_{\mathbb{T}^3}W(u_n)dx + \frac{1}{\eta_n^5}||u_n - ⨍ u_n||^2_{H^{-1}}.$$

Furthermore, define the continuous and strictly increasing function

$$\phi(s) := 2\int_0^s \sqrt{W(t)}dt.$$

Then, there exist

$$E_{\varepsilon_n,\eta_n}(v_n) \geq \frac{1}{\eta_n^2}\int_{\mathbb{T}^3}|\nabla\phi(u_n)| + \frac{1}{\eta_n^5}||u_n - ⨍ u_n||^2_{H^{-1}} \tag{2.5.8}$$

since the consequence of the inequality $a^2 + b^2 \geq 2ab$.

In addition, define $\alpha_n = 1/(\sigma - \eta_n^\xi)$. Recall the previous surface tension $\sigma := 2\int_0^1 \sqrt{W(t)}dt = \phi(1) - \phi(0)$. Thus,

$$\phi(1 - 2\delta_n) - \phi(2\delta_n) = \phi(1) - \phi(0) - \eta^x i_n = \frac{1}{\alpha_n},$$

where the quadratic behaviour of W at 0 and 1 implies that $\delta_n = O(\eta_n^{\xi/2})$. Therefore, $\delta_n > 0$. Besides, the notation$[n]$ for the clipping to the interval $[0, 1]$ is defined

$$[u] := \min\{1, \max\{0, u\}\}.$$

The size of the set

$$A_n := \left\{ t \in [\phi(0), \phi(1)] : \mathcal{H}^1(\partial^* \{\phi([u_n]) > t\} \geq \alpha_n \int_{\mathbb{T}^3} |\nabla \phi([u_n])| \right\}$$

can be estimated by

$$|A_n| = \int_{A_n} 1dt \leq \frac{1}{\alpha_n \int_{\mathbb{T}^3} |\nabla \phi([u_n])|} \int_{\phi(0)}^{\phi(1)} \mathcal{H}^1(\partial^* \{\phi([u_n]) > t\})dt = \frac{1}{\alpha_n}.$$

by using the characterization of perimeter from Theorem 2.1 in [24] as

$$\int_{\mathbb{T}^3} |\nabla \phi([u_n])| = \int_{\phi(0)}^{\phi(1)} \mathcal{H}^1(\partial^* \{\phi([u_n]) > t\})dt.$$

Therefore,

$$\mathcal{H}^1(\partial^* \{\phi([u_n]) > t_n\}) \leq \alpha_n \int_{\mathbb{T}^3} |\nabla \phi([u_n])| \tag{2.5.9}$$

where $t_n \in [\phi(\delta_n), \phi(1 - \phi_n)]/A_n$ by the definition of α_n and δ_n.

In addition, define an auxiliary sequence $\overline{u}_n$. Its corresponding $\overline{v}_n = \overline{u}_n/\eta_n$ is admissible for the sharp-interface fucntional E_η. Then, map the values of u_n to $\{0, 1\}$ with cut off

$\phi^{-1}(t_n)$ as

$$\overline{u}_n(x) := \begin{cases} 0 & \text{if } \phi(u_n(x)) < t_n \\ 1 & \text{if } \phi(u_n(x)) \geq t_n \end{cases}$$

Therefore, there exists

$$\int |\nabla \overline{u}_n| = \mathcal{H}^1(\partial^* \{\phi([u_n]) > t_n\}). \tag{2.5.10}$$

Then, define ψ as

$$\psi_n(u) := \begin{cases} u^2 & \text{if } \phi((u)) < t_n \\ (1-u)^2 & \text{if } \phi((u)) \geq t_n \end{cases}$$

since $\phi^{-1}(t_n) \in [\delta_n, 1-\phi_n]$. Also,

$$\psi_n(u) \leq C\delta_n^{-2} W(u) \leq C' \delta_n^{-\xi} W(u)$$

for some C and C' independent of n, which means that $\psi_n(u)$ is bounded by an increasing factor times W. Thus, the sequences $\overline{u}_n$ and u_n are close in L^2 and the final estimate results from the boundedness of $E_{\varepsilon_n,\eta_n}(v_n)$ is

$$||u_n - \overline{u}_n|| = \int_{\mathbb{T}^3} \psi_n(u_n) \leq C' \eta_n^{-\xi} \int_{\mathbb{T}^3} W(u_n) = O(\varepsilon_n \eta^{2-\xi}) \to 0.$$

Furthermore, they are close in H^{-1} as

$$\begin{aligned} ||u_n - \overline{u}_n - \fint (u_n - \overline{u}_n)||_{H^{-1}} &\leq C||u_n - \overline{u}_n - \fint (u_n - \overline{u}_n)||_{L^2} \\ &\leq ||u_n - \overline{u}_n||_{L^2} \\ &= O(\varepsilon_n^{1/2} \eta_n^{(1-\xi/2)}) \to 0. \end{aligned} \tag{2.5.11}$$

For the squared norms, the same holds and using the hypothesis $\varepsilon_n = o(\eta_n^{4+\xi})$ as

$$
\begin{aligned}
&\Big| ||u_n - \fint u_n||^2_{H^{-1}} - ||\overline{u}_n - \fint \overline{u}_n||^2_{H^{-1}} \Big| \\
&\leq \Big(||u_n - \fint u_n||_{H^{-1}} + ||\overline{u}_n - \fint \overline{u}_n||_{H^{-1}} \Big) ||u_n - \overline{u}_n - \fint (u_n - \overline{u}_n)||_{H^{-1}} \\
&\leq \Big(||u_n - \fint u_n||_{H^{-1}} + ||u_n - \overline{u}_n - \fint (u_n - \overline{u}_n)||_{H^{-1}} \Big) O(\varepsilon_n^{1/2} \eta_n^{(1-\xi/2)}) \\
&= (\eta_n^5 E_{\varepsilon_n,\eta_n}(v_n))^{1/2} O(\varepsilon_n^{1/2} \eta_n^{(1-\xi/2)}) + O(\varepsilon_n \eta_n^{2-\xi}) \\
&= O(\varepsilon_n^{1/2} \eta_n^{(7/2-\xi/2)}) + O(\varepsilon_n \eta_n^{2-\xi}) \\
&= o(\eta_n^6).
\end{aligned}
\tag{2.5.12}
$$

As a result, the lower bound (2.5.8) in the sequence $\overline{u}_n$ by (2.5.9) and (2.5.10) is

$$
\begin{aligned}
E_{\varepsilon_n,\eta_n}(v_n) &\geq \frac{1}{\alpha_n \eta_n^2} \mathcal{H}^1(\partial^* \{\phi([u_n]) > t_n\}) + \frac{1}{\eta_n^5} ||u_n - \fint u_n||^2_{H^{-1}} \ \Big(\text{by } (2.5.8), (2.5.9)\Big) \\
&= \frac{1}{\alpha_n \eta_n^2} \int_{\mathbb{T}^3} |\nabla \overline{u}_n| + \frac{1}{\eta_n^5} ||\overline{u}_n - \fint \overline{u}_n||^2_{H^{-1}} + o(\eta_n) \ \Big(\text{by } (2.5.10), (2.5.12)\Big) \\
&= \frac{\eta_n}{\alpha_n} \int_{\mathbb{T}^3} |\nabla \overline{v}_n| + \eta_n ||\overline{v}_n - \fint \overline{v}_n||^2_{H^{-1}} + o(\eta_n) \\
&\geq \frac{1}{\sigma \alpha_n} \mathsf{E}_{\eta_n}(\overline{v}_n) + o(\eta_n) \ \Big(\text{since } \sigma\alpha_n > 1 \ (\sigma\alpha_n \to 1 \text{ as } n \to \infty)\Big)
\end{aligned}
\tag{2.5.13}
$$

Moreover, by combining the above results and Theorem 2.4.4, there exists a subsequence $\overline{v}_{n_k}$ converging to a limit v_0 with countable support as

$$
\liminf_{k \to \infty} \mathsf{E}_{\eta_{n_k}}(\overline{v}_{n_k}) \geq \mathsf{E}_0(v_0). \tag{2.5.14}
$$

Besides, since $\varphi \in C(\mathbb{T}^3)$, this subsequence converges weakly to the same limit as

$$
\Big| \int_{\mathbb{T}^3} (v_{n_k} - \overline{v}_{n_k}) \phi \Big| \leq \frac{1}{\eta_{n_k}^3} ||(u_{n_k} - \overline{u}_{n_k})||_{L^2} ||\varphi||_{L^2} = O(\varepsilon_{n_k}^{1/2} \eta_{n_k}^{-2-\xi/2}) \to 0.
$$

Therefore, the compactness of the sequence v_n and the characterization of the support of the limit are proved as the lower bound inequality (2.4.3) is proved by (2.5.13) and (2.5.14).

For the lower bound of $F_{\varepsilon,\eta}$, the boundedness of F_{ε_n,η_n} implies the boundedness of E_{ε_n,η_n}. Thus, the characterization also implies. As a result, by (2.5.13),

$$\begin{aligned} F_{\varepsilon_n,\eta_n} &= \frac{1}{\eta_n}\Big[E_{\varepsilon_n,\eta_n}(v_n) - \overline{e}_0\Big(\int_{\mathbb{T}}^3 v_n\Big)\Big] \\ &\geq \frac{1}{\eta_n}\Big[\mathsf{E}_{\eta_n}(\overline{v}_n) - \overline{e}_0\Big(\int_{\mathbb{T}}^3 v_n\Big)\Big] + \frac{1}{\eta_n}\Big(\frac{1}{\sigma\alpha_n} - 1\Big)\mathsf{E}_{\eta_n}(\overline{v}_n) + o(1). \end{aligned}$$

As a result, the lower bound for F_η ipmlies

$$\liminf_{n\to\infty} F_{\varepsilon_n,\eta_n}(v_n) \geq \mathsf{F}_0(v_0),$$

since $\sigma\alpha_n = 1 + o(\eta_n^\xi)$ with $\xi > 1$. Thus, (2.4.6) is proved.

Upper Bound

For the upper bound, at first, it is needed to deal with $E_{\varepsilon,\eta}$. By the proof of Theorem 2.4.4, there exists the v_0 as

$$v_0 = \sum_{i=1}^{N} m^i \delta_{x^i},\ x^i \text{ distinct.}$$

Thus, there exists a sequence $\overline{v}_n \to v_0$ by (2.4.13) such that

$$\limsup_{n\to\infty} E_{\varepsilon_n,\eta_n}(\overline{v}_n) \leq \mathsf{E}_0(v_0). \tag{2.5.15}$$

Then, for E_η such that an admissible sequence $v_n \to v_0$ with a given v_0,

$$\lim_{n\to\infty} E_{\eta_n}(v_n) = \mathsf{E}_0(v_0). \tag{2.5.16}$$

Moreover, define the characteristic function of a subset of $\mathbb{T}^3$ composed of N sets as

$$u_n := \eta_n^3 v_n,$$

where diameters are decreasing to zero. By Lemma 2.5.3 ($\alpha = \eta_n$), there exsits a $C_0(\eta_n)$ for any $\varepsilon_n > 0$ with each n. Therefore, there exists an approximation as

$$\int_{\mathbb{T}^3} \varepsilon_n |\nabla \overline{u}_n|^2 + \frac{1}{\varepsilon_n} \overline{u}_n^2 (1 - \overline{u}_n^2) dx \leq (\sigma + \eta_n) \int_{\mathbb{T}^3} |\nabla u_n| \tag{2.5.17}$$

and

$$\int_{\mathbb{T}^3} |\overline{u}_n - u_n| dx \leq C_0(\eta_n) \varepsilon_n \int_{\mathbb{T}^3} |\nabla u_n|$$

for $\overline{u}_n \in H^1(\mathbb{T}^3, [0,1])$. In addition, define

$$\overline{v}_n = \frac{\overline{u}_n}{\eta_n^3}.$$

Thus,

$$\begin{aligned} ||\overline{v}_n - v_n||_{L^1(\mathbb{T}^3)} &= \int_{\mathbb{T}^3} |\overline{u}_n - u_n| dx \\ &\leq \frac{C_0(\eta_n)\varepsilon_n}{\eta_n^3} \int_{\mathbb{T}^3} |\nabla u_n| \\ &\leq C \frac{C_0(\eta_n)\varepsilon_n}{\eta_n}. \end{aligned} \tag{2.5.18}$$

Then, the estimated H^{-1}-norm is

$$\begin{aligned} ||v_n - \overline{v}_n - \fint (v_n - \overline{v}_n)||_{H^{-1}(\mathbb{T}^3)} &\leq C||v_n - \overline{v}_n - \fint (v_n - \overline{v}_n)||_{L^2(\mathbb{T}^3)} \\ &\leq C||v_n - \overline{v}_n||^2_{L^2(\mathbb{T}^3)} \\ &\leq C||v_n - \overline{v}_n||_{L^\infty(\mathbb{T}^3)} ||v_n - \overline{v}_n||_{L^1(\mathbb{T}^3)} \\ &\leq C \frac{C_0(\eta_n)\varepsilon_n}{\eta_n^4}. \end{aligned} \tag{2.5.19}$$

Then,

$$\begin{aligned}
E_{\varepsilon_n,\eta_n}(\overline{v}_n) &= \frac{\varepsilon_n}{\eta_n^2}\int_{\mathbb{T}^3}|\nabla\overline{u}_n|^2 + \frac{1}{\eta_n^2\varepsilon_n}W(\overline{u}_n) + \frac{1}{\eta_n^5}||\overline{u}_n - \fint \overline{u}_n||_{H^{-1}}^2 \\
&= \frac{1}{\eta_n^2}\int_{\mathbb{T}^3}\Big(\varepsilon_n|\nabla\overline{u}_n|^2 + \frac{1}{\varepsilon_n}\overline{u}_n^2(1-\overline{u}_n^2)\Big)dx + \eta_n||\overline{v}_n - \fint \overline{v}_n||_{H^{-1}}^2 \\
&\le \frac{1}{\eta_n^2}\int_{\mathbb{T}^3}\Big(\varepsilon_n|\nabla\overline{u}_n|^2 + \frac{1}{\varepsilon_n}\overline{u}_n^2(1-\overline{u}_n^2)\Big)dx + \eta_n||v_n - \fint v_n||_{H^{-1}}^2 \\
&\qquad + ||v_n - \overline{v}_n - \fint (v_n - \overline{v}_n)||_{H^{-1}(\mathbb{T}^3)} \\
&\le \eta_n(\sigma+\eta_n)\int_{\mathbb{T}^3}|\nabla v_n| + \eta_n||v_n - \fint v_n||_{H^{-1}}^2 + C\frac{C_0(\eta_n)\varepsilon_n}{\eta_n^3} \ \Big(\text{by (2.5.17) and (2.5.19)}\Big) \\
&= \mathsf{E}_\eta(v_n) + \eta_n^2\int_{\mathbb{T}^3}|\nabla v_n| + C\frac{C_0(\eta_n)\varepsilon_n}{\eta_n^3}.
\end{aligned} \tag{2.5.20}$$

Then, define a function C_1 as in the Theorem [18]. Therefore,

$$C\frac{C_0(\eta_n)\varepsilon_n}{\eta_n^3} \to 0 \quad \text{as} \quad n \to 0 \tag{2.5.21}$$

such that satisfies $\varepsilon_n \le C_1(\eta_n)$. As a result, by taking the lim sup as $n \to \infty$ in (2.5.20) and (2.5.16),

$$\limsup_{n\to\infty} E_{\varepsilon_n,\eta_n}(\overline{v}_n) \le \mathsf{E}_0(v_0).$$

Moreover, define

$$v_0 = \sum_{i=1}^{N} m^i\delta_{x^i},\ \{m^i\} \in \mathcal{M}.$$

Thus, there exists a sequence $v_n \to v_0$ by (2.4.14) such that

$$\lim_{n\to\infty} F_{\eta_n}(v_n) = \mathsf{F}(v_0). \tag{2.5.22}$$

Follow the steps in (2.5.20), there exists

$$E_{\varepsilon_n,\eta_n}(\overline{v}_n) \le \mathsf{E}_\eta(v_n) + \eta_n^3\int_{\mathbb{T}^3}|\nabla v_n| + C\frac{C_0(\eta_n^2)\varepsilon_n}{\eta_n^3}. \tag{2.5.23}$$

Besides, let L be the local Lipschitz constant of e_0, there exists

$$\begin{aligned}
F_{\varepsilon_n,\eta_n}(\overline{v}_n) &= \eta_n^{-1}\Big[E_{\varepsilon,\eta}(\overline{v}_n) - e_0\Big(\int_{\mathbb{T}^3}\overline{v}_n\Big)\Big] \\
&\le \eta_n^{-1}\Big[\mathsf{E}_\eta(v_n) + \eta_n^3\int_{\mathbb{T}^3}|\nabla v_n| + C\frac{C_0(\eta_n^2)\varepsilon_n}{\eta_n^3} \\
&\qquad - e_0\Big(\int_{\mathbb{T}^3}v_n\Big) + \Big(e_0\Big(\int_{\mathbb{T}^3}v_n\Big) - e_0\Big(\int_{\mathbb{T}^3}\overline{v}_n\Big)\Big)\Big] \ \Big(\text{by } (2.5.23)\Big) \\
&\le F_{\eta_n}(v_n) + O(\eta_n) + \eta_n^{-1}\Big[L||\overline{v}_n - v_n||_{L^{-1}} + C\frac{C_0(\eta_n^2)\varepsilon_n}{\eta_n^3}\Big].
\end{aligned}$$

Then, define a function C_2 as in the Theorem. Therefore,

$$C\frac{C_0(\eta_n^2)\varepsilon_n}{\eta_n^3} \to 0 \quad \text{as} \quad n \to 0 \tag{2.5.24}$$

such that satisfies $\varepsilon_n \le C_2(\eta_n)$. As a result,

$$\limsup_{n\to\infty} F_{\varepsilon_n,\eta_n}(\overline{v}_n) \le \mathsf{F}_0(v_0).$$

□

3 The Non-Local Isoperimetric Problem Under Confinement

3.1 Introduction

The droplet regime in the spare A-phase is described effectively by the droplet centers, which are the particles in the previous study. The droplet regime is in the mass fraction between the two phases tends to zero with very strong nonlocal interactions [2]. Besides, the minimizing phases resemble small spherical inclusion of one phase in a large sea of the second phase. The droplet regime is able to decompose the nonlocal effects into self-effects as a single droplet and interaction-effects between different particles. Furthermore, since replacing the torus $\mathbb{T}^3$ with the smooth bounded domain $\mathcal{D} \in \mathbb{R}^3$ is reasonable in mathematics and physics, there is no need to impose periodic boundary conditions to observe the droplet splitting and confinement. In the previous study, the location of the centers x_η^i is determined by the next order term in the energy expansion and a Coulomb-like repulsion will arise from the nonlocal term. On the same definitions, define the energy functional

$$\mathsf{E}_\eta(v) := \eta \int_{\mathbb{T}^3} |\nabla v| + \eta ||v - M||^2_{H^{-1}(\mathbb{T}^3)} - \int_{\mathbb{T}^3} v(x)\rho(x)dx \tag{3.1.1}$$

for a fixed function $\rho \in C(\mathbb{T}^3)$ in the limit of $\eta \to 0$ by rescaling the energy functional from previous study. The confinement term here is at the level of particle interaction. It is drawing the centers x^i_η towards the global maximum of the nanoparticle density $\rho(x)$. Therefore, isolate individual droplets at a much smaller scale is necessary.

Suppose minimizers form n droplets and center at points $x^i_n = \delta p_i, i = 1, 2, ..., n; \delta = \delta(\eta) \to 0$. Therefore, define

$$\nu_\eta := \sum_{i=1}^{n} \eta^{-3} w_i \Big(\frac{x - \delta p_i}{\eta} \Big),\ w_i \in BV(\mathbb{R}^3; \{0, 1\}) \tag{3.1.2}$$

as an admissible test configuration. Define

$$m^i := \int_{\mathbb{R}^3} w_i dx.$$

Then, by expressing the H^{-1}-norm in Green's function ($G(x, y) \sim 1/4\pi|x - y|$), evaluate $\mathsf{E}_\eta(\nu_\eta)$ asymptotically as

$$\begin{aligned} \mathsf{E}_\eta(\nu_\eta) \simeq & \sum_{i=1}^{n} \Big[\int_{\mathbb{R}^3} |\nabla w_i| + \eta ||w_i||^2_{H^{-1}(\mathbb{R}^3)} \Big] - M\rho_{\max} \\ & + \left[\frac{\eta}{\delta} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{m^i m^j}{4\pi |p_i - p_j|} + \delta^2 \sum_{i=1}^{n} m^i q(p_i) \right], \end{aligned} \tag{3.1.3}$$

where the distance between droplet centers $\delta(\eta) \gg \eta$.

Recall (2.4.1), since droplet profiles z_i are minimizers for the nonlocal isoperimetric problem in $\mathbb{R}^3$, then define

$$e_0(m) := \inf\left\{\int_{\mathbb{R}^3} |\nabla z| + ||z||^2_{H^{-1}(\mathbb{R}^3)} : z \in BV(\mathbb{R}^3;\{0,1\}), \int_{\mathbb{R}^3} z = m\right\}. \qquad (3.1.4)$$

Therefore, for minimizers of E_η in Gamow functional, define

$$\mathcal{M}_0 := \{m > 0 : e_0(m) \text{ admits a minimizer}\},$$

for which the nonlocal isoperimetric probelm attains a minimizer. When total mass $M \in \mathcal{M}_0$, there is no need to split and minimizers v_η remain connected as $\eta \to 0$. However, when total mass $M \notin \mathcal{M}_0$, minimizers will split into droplets with mass $m^i \in \mathcal{M}_0$. Here is an optimal droplet blowup. As a result, recall (2.4.3), define a set

$$\mathcal{M}_1 := \left\{\{m^i\}_{i=1}^n : n \in \mathbb{N}, m^i \geq 0, \sum_{i=1}^n m^i = M, \text{ such that } e_0(m^i) \text{ admits a minimizer for each } i\right\}$$

where droplet masses must lie in.

3.2 Main Results

The most important result is to confirm the expected behavior by means of a precise asymptotic expansion of the energy of minimizers.

Theorem 3.2.1. *Let v_η be minimizers of E_η in $BV(\mathbb{T}^3, \{0, \eta^{-3}\})$. Since $m^i := \int_{\mathbb{R}^3} w_i dx$, $\int_{\mathbb{R}^3} v_\eta dx = M$.*

- *(I) Supp $v_\eta \subset B_r(0) \subset \mathbb{T}^3$ for all sufficiently small $\eta > 0$ and any $r > 0$.*
- *(II) There exists a subsequence of $\eta \to 0$ and points $y_n \in \mathbb{T}^3$ with $|y_n| \leq C\eta^{1/2}$ for $M \in \mathcal{M}_0$ such that*

  $$v_\eta - \eta^{-3} z_M\Big(\frac{x - y_n}{\eta}\Big) \to 0 \in L^1(\mathbb{T}^3),$$

  where z_M attains the minimum $e_0(M)$.

- (III) There exits a subsequence of $\eta \to 0, n \in \mathbb{N}, \{m^i\}_{i=1}^n \in \mathcal{M}_1$, and distinct points $\{x_\eta^1, ..., x_\eta^n\}$ for $M \notin \mathcal{M}_0$ such that

$$v_\eta - \sum_{i=1}^{n} \eta^{-3} w_i \Big(\frac{x - x_\eta^i}{\eta} \Big) \to 0 \in L^1(\mathbb{T}^3),\ w_i \in BV(\mathbb{T}^3, \{0,1\}),$$

where w_i attains the minimum in $e_0(m_i), i = 1, ..., n$. Furthermore,

$$\eta^{-1/3} x_\eta^i \to x_i \in \mathbb{R}^3$$

and

$$\mathsf{E}_\eta(v_\eta) = e_0(M) - M\rho_{\max} + \eta^{2/3} \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{m^i m^j}{4\pi |x_i - x_j|} + \sum_{i=1}^{n} m^i q(x_i) \right] + o(\eta^{2/3}).$$

In addition, the expression in brackets above is minimized by the choice of points $\{x_1, ..., x_n\}$ given the values $\{m^i\}_{i=1}^n \in \mathcal{M}_1$.

3.3 Structure of Minimizers

For studying the concentration structure of minimizers, will begin with second-order approximation since the first-order limit functional and convergence result have been studied already in the previous section. This limit depends on the specific form of the penalizing measure ρ. Mass constrained minimizers of E_η concentrate at the origin as $\eta \to 0$, which depends on the size of the mass constraint M. When M is very large, a minimizer of e_0 fails to exist and the minimizer split at a scale larger than η as $\eta \to 0$. When M is very small, there is no splitting.

Moreover, consider the upper bounds on the minimum energy of E_η.

Lemma 3.3.1. For any $n \in \mathbb{N}$, $\{p_i\}_{i=1}^n$ distinct fixed points in $\mathbb{R}^3$ and $\{m^i\}_{i=1}^n \in \mathcal{M}_1$. Thus,

$$\min_{\int_{\mathbb{T}^3} v = M} \mathsf{E}_\eta(v) \leq (e_0(M) - M\rho_{\max}) + \eta^{2/3} \left[\sum_{i=1}^{n} m^i q(p_i) + \frac{1}{4\pi} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{m^i m^j}{|p_i - p_j|} \right] + o(\eta^{2/3}). \quad (3.3.1)$$

Besides, when $M \notin \mathcal{M}_0$,

$$\min_{\int_{\mathbb{T}^3} v dx = M} \mathsf{E}_\eta(v) \leq (e_0(M) - M\rho_{\max}) + O(\eta). \tag{3.3.2}$$

As a result, the upper bound is verified. Furthermore, normally, the most difficult part is to verify the lower bound. First, establish the existence of points in $\mathbb{T}^3$ by a compactness result. These points are separated by a scale much larger than η apart. Therefore, the weighted Dirac-Delta measure at these points approximate v_η. Then, it achieves the minimum of the first-order energy E_0 by the existence of components of supp v_η, since the supports are η-resaclings of minimizers of the nonlocal isoperimetric problem. Moreover, consider that there can only be finitely many distinct components for minimizers of E_η, there is a strong convergence result and there may be a unique component.

Lemma 3.3.2. For each $\eta > 0$, let v_η be a minimizer of E_η with $\int_{\mathbb{T}^3} v_\eta dx = M$. Therefore, there exists a subsequence $\eta \to 0, n \in \mathbb{N}, \{m^i\}_{i=1}^n \subset (0, \infty), \{x_\eta^i\}_{i=1}^n \subset \mathbb{T}^3$ and functions $w_\eta^i \in BV(\mathbb{T}^3; \{0, 1/\eta^3\})$ with $||w_\eta^i||_{L^1(\mathbb{T}^3} = m^i + o(1)$ as $\eta \to 0$ for $n \geq 2$ such

$$\frac{|x_\eta^i - x_\eta^j|}{\eta} \to \infty \text{ for every } i \neq j; \tag{3.3.3}$$

$$\Big\| v_\eta - \sum_{i=1}^n w_\eta^i \Big\|_{L^1(\mathbb{T}^3)} \to 0; \tag{3.3.4}$$

$$e_0(m^i) \text{ is attained for each } i = 1, 2, ..., n \text{ and } e_0(M) = \sum_{i=1}^n e_0(m^i); \tag{3.3.5}$$

$$\liminf_{\eta \to 0} \mathsf{E}_\eta(v_\eta) \geq \liminf_{\eta \to 0} \sum_{i=1}^n \mathsf{E}_\eta(w_\eta^i) \geq \sum_{i=1}^n e_0(m^i) - M\rho_{\max}. \tag{3.3.6}$$

Besides, when $n = 1$, then $M \in \mathcal{M}_0$. Thus, there exist points $x_\eta \in \mathbb{T}^3$ such that

$$v_\eta - \eta^{-3} z_M \Big(\frac{x - x_\eta}{\eta} \to 0 \Big) \text{ in } L^1(\mathbb{T}^3) \text{ as } \eta \to 0, z_M \text{ attains the minimum } e_0(M). \tag{3.3.7}$$

For the detailed proof, please see [Lemma 2.5 [2]].

3.4 Regularity of Minimizers

In addition, there exists a possible error of $o(1)$. It means it is not sharp enough to compute the interaction between droplets. Therefore, it is necessary to use the regularity theory of minimizers to refine the decomposition to obtain a lower bound without error term. The fundamental idea is to show the minimizers v_η of E_η blow up to $\omega-$minimizers of perimeter in $\mathbb{R}^3$ which ω is independent of η.

By defining the $\omega-$minimality, there exists a strong regularity result for minimizers v_η. Let $\mathcal{O} \subset \mathbb{R}^3$ be an open set and $w > 0$. Let A be a set of finite perimeter as $A \subset \mathbb{R}^3$. Let B be any set of finite perimeter as $B \subset \mathbb{R}^3$ any ball $B_r(x_0) \subset \mathcal{O}$. Thus, if $A \triangle B \subset\subset B_r(x_0)$, then A is an ω-minimizer for the perimeter functional $\int_{\mathbb{R}^3} |\nabla \chi_A|$ in $\mathcal{O}$. Therefore, there exists

$$\int_{\mathcal{O}} |\nabla \chi_A| \leq \int_{\mathcal{O}} |\nabla \chi_B| + wr^3.$$

In addition, the second step is to blow up the minimziers set $A_\eta = \text{supp } v_\eta$. For any fixed $p \in \mathbb{T}^3$ and a $R > 0$ which is given in [Lemma 2.8 [2]], there exists the $\widetilde{A}_{p,\eta}$ as an ω-minimizer of perimeter in $B_R(0)$ for $\omega > 0$, and uniformly for all $\eta \in (0, \frac{1}{2R})$. Besides, by [25], define an unconstrained functional

$$\mathsf{E}^\lambda_\eta(u) := \mathsf{E}_\eta(u) + \lambda \Big| \int_{\mathbb{T}^3} u dx - M \Big|,$$

for any $\lambda > 0$ and $u \in BV(\mathbb{T}^3; \{0,1\}$. This functional will penalizes deviations from the usual mass constraint. Therefore, there exists constants $\eta_0, \lambda_0 > 0$ such that for every $0 < \eta < \eta_0$

$$\inf\{\mathsf{E}^{\lambda_0}_\eta(u) : u \in BV(\mathbb{T}^3; \{0,1\})\} = \mathsf{E}^{\lambda_0}_\eta(v_\eta) = \mathsf{E}_\eta(v_\eta).$$

Therefore, the following regularity result is the important consequence of $\omega-$minimiality.

Theorem 3.4.1. Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded open set, and $\widehat{\Omega}_\eta \subset \mathcal{O}$ be a sequence of

$\omega-$minimizers of the perimeter functional such that

$$\sum_{\eta>0}\int_{\mathcal{O}}|\nabla\chi_{\widehat{\Omega}_\eta}|<+\infty \quad \text{and} \quad \chi_{\widehat{\Omega}_\eta}\to\chi_\Omega \text{ in } L^1(\mathcal{O})$$

for $\Omega\subset\mathcal{O}$ of class C^2. Then, for the small enough η, $\widehat{\Omega}_\eta$ is of class $C^{1,1/2}$ and

$$\partial\widehat{\Omega}_\eta=\{x+\psi_\eta(x)\nu^\Omega(x):x\in\partial\Omega\}$$

with $\psi_\eta\to 0$ in $C^{1,\alpha}(\partial\Omega)$ for all $\alpha\in(0,1/2)$ where ν^Ω denotes the unit outward normal to $\partial\Omega$.

As a result, by the regularity theorem to minimizers of E_η, the minimizers split exactly and disjointly into the sets U_η^i found in Lemma 2.5 of [2].

Lemma 3.4.2. Let $v_\eta=\eta^{-3}\chi_{A_\eta}$ be minimizers of E_η with $|A_\eta|=M\eta^3$. Let $n\in\mathbb{N}$, and u_η^i and $U_\eta^i\in\mathbb{R}^3, i=1,2,...,n$. There exists $R>0$ independent of η such that

$$U_\eta^i\in B_R(0),\quad \sum_{i=1}^n|U_\eta^i|=\sum_{i=1}^n m_\eta^i=M,\quad \text{and } v_\eta(x)=\sum_{i=1}^n u_\eta^i(x).$$

for all sufficiently small $\eta>0$. Moreover, the sharp lower bound on the energy is

$$\mathsf{E}_\eta(v_\eta)\geq\sum_{i=1}^n\mathsf{E}_\eta(u_\eta^i)+\sum_{\substack{i,j=1\\ i\neq j}}^n\eta\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}\frac{u_\eta^i u_\eta^j}{4\pi|x-y|}dxdy+\mathcal{O}(\eta).$$

for sufficiently small $\eta>0$.

v_η splits into exactly n components and the residual sets exactly vanish. Thus, $\Xi_\eta, V_\eta^n=\emptyset$ for small η.

3.5 Proof

To prove the main result - Theorem 3.2.1, one good way is to match the upper bounds and lower bounds on $\mathsf{E}_\eta(v_\eta)$. The sharp form of the decomposition is used to it.

Upper Bound

Let $n \in \mathbb{N}$, $\{p_i\}_{i=1}^n$ distinct fixed points in $\mathbb{R}^3$ and $\{m^i\}_{i=1}^n \in \mathcal{M}_1$. Recall (3.1.4), then there exists z^i attains the minimum in $e_0(m^i)$. In addition, there exists a constant $r > 0$ for $Z^i :=$ supp $z^i \subset B_r(0), i = 1, 2, ..., n$ by considering a finite number of z^i only since the support of z_i are bounded. Then, define

$$\nu_\eta^i := \frac{1}{\eta^3} z^i \Big(\frac{x - \eta^{1/3} p_i}{\eta} \Big)$$

and

$$\nu_\eta := \sum_{i=1}^n \nu_\eta^i.$$

Here, ν_η is a function on $\mathbb{T}^3$ since supp$\nu_\eta \subset B_{1/4}(0)$ for all very small $\eta > 0$. Therefore,

$$\begin{aligned} \mathsf{E}_\eta(\nu_\eta) &= \sum_{i=1}^n \eta \int_{\mathbb{T}^3} |\nabla \nu_\eta| + \eta \sum_{i,j=1}^n \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \nu_\eta^i(x) \nu_\eta^j(y) G(x-y) dx dy - \sum_{i=1}^n \int_{\mathbb{T}^3} \nu_\eta^i(x) \rho(x) dx \\ &= \sum_{i=1}^n e_0(m_i) - M\rho_{\max} + \eta \sum_{\substack{i,j=1 \\ i \neq j}}^n \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \frac{\nu_\eta^i(x) \nu_\eta^j(y)}{4\pi |x-y|} dx dy \\ &\quad + \sum_{i=1}^n \int_{\mathbb{T}^3} \nu_\eta^i(x) (\rho_{\max} - \rho(x)) dx + O(\eta). \end{aligned} \tag{3.5.1}$$

Moreover, set the change of variables as

$$\eta\xi = x - \eta^{1/3} p_i \quad \text{and} \quad \eta\zeta = y - \eta^{1/3} p_j.$$

Thus, to evaluate $i \neq j$, by the Dominated Convergence applied to the integral ($\eta \to 0$),

$$\begin{aligned} \eta \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \frac{\nu_\eta^i(x) \nu_\eta^j(y)}{4\pi |x-y|} dx dy &= \eta^{2/3} \int_{Z^i} \int_{Z^j} \frac{1}{4\pi |\eta^{2/3}(\xi - \zeta) - (p_i - p_j)|} d\xi d\zeta \\ &= \eta^{2/3} \frac{m^i m^j}{4\pi |p_i - p_j|} + o(\eta^{2/3}). \end{aligned} \tag{3.5.2}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{T}^3} \nu_\eta^i(x)(\rho_{\max} - \rho(x))dx &= \int_{Z^i} q\Big(\eta^{1/3}[p_i + \eta^{2/3}\xi]\Big)d\xi + o(\eta^{2/3}) \\ &= \eta^{2/3}\int_{Z^i} q(p_i + \eta^{2/3}\xi)d\xi + o(\eta^{2/3}) \\ &= \eta^{2/3} m^i q(p_i) + o(\eta^{2/3}).\end{aligned} \tag{3.5.3}$$

As a result, the desired upped bound for $n \geq 2$ is by inserting (3.5.2) and (3.5.3) into (3.5.1).

When $M \in \mathcal{M}_0$, by attaining $e_0(M)$, let $n = 1, p_1 = 0$, and $z = \chi_Z$. Then, define $\nu_\eta(x) = z(x/\eta)$. Therefore, (3.5.1) can be written as

$$\begin{aligned}\mathsf{E}_\eta(\nu_\eta) &= e_0(M) - M\rho_{\max} + \int_Z [\rho(\eta\xi) - \rho_{\max}]d\xi + O(\eta) \\ &\leq e_0(M) - M\rho_{\max} + O(\eta).\end{aligned}$$

Note that the principal error comes from the regular part of the Green's function.

Lower Bound

To derive the lower bound, it is appropriate to begin with the droplet centers $\{x_\eta^i\}$. For each $\eta > 0$, let v_η be a minimizer of E_η with $\int_{\mathbb{T}^3} v_\eta dx = M$. Define

$$\lambda_\eta := \min_{i \neq j} |x_\eta^i - x_\eta^j| \quad \text{and} \quad \beta_\eta := \max_i |x_\eta^i| \tag{3.5.4}$$

for a finite number of $i, j = 1, 2, ..., n$. Let $\{u^i - \eta\}_{i=1}^n$ with $v_\eta = \sum_{i=1}^n u_\eta^i$ be the functions found in Lemma 3.3.2. Thus, $\beta_\eta \gg \eta$ since $|x_\eta^i - x_\eta^j| \gg \eta$.

The first step is to evaluate the nonlocal and confinement terms. Consider the case $n \geq 2$. There exist

$$\left| \frac{1}{|x-y|} - \frac{1}{|x_\eta^i - x_\eta^j|} \right| \leq \frac{4R_\eta}{|x_\eta^i - x_\eta^j|^2}$$

for all $x \in$ supp $u_\eta^i = (\eta U_\eta^i + x_\eta^i)$ and $y \in$ supp $u_\eta^j = (\eta U_\eta^j + x_\eta^j)$, $i \neq j$ since $U_\eta^i \subset B_R(0)$.

Therefore,

$$\int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \frac{u_\eta^i u_\eta^j}{4\pi|x-y|}dxdy = \frac{m_\eta^i m_\eta^j}{4\pi|x_\eta^i - x_\eta^j|} + O(\eta|x_\eta^i - x_\eta^j|^{-2}) = \frac{m_\eta^i m_\eta^j}{4\pi|x_\eta^i - x_\eta^j|}(1-o(1)). \quad (3.5.5)$$

For the confinement term, set a k in $|x_\eta^k = \beta_\eta$. Thus, there exits $|x| \geq 1/2\beta_\eta$ for $x \in (\eta U_\eta^k + x_\eta^k)$. Therefore, the rough estimate is

$$\rho(x) - \rho_{\max} \leq -q(x) + o(|x|^2) \leq -c_2\beta_\eta^2, \text{ constant } c_2 > 0 \text{ and independent of } \eta.$$

by using the hypothesis on the structure of $\rho(x)$ near zero. As a result, a rough lower bound is

$$\begin{aligned} \mathsf{E}_\eta(v_\eta) &\geq \sum_{i=1}^n (e_0(m_\eta^i) - m_\eta^i \rho_{\max}) + \sum_{\substack{i,j=1 \\ i\neq j}}^n \eta \frac{m^i m^j}{4\pi|x_\eta^i - x_\eta^j|}(1-o(1)) + c_1 m_\eta^k \beta_\eta^2 \\ &\geq e_0(M) - M\rho_{\max} + \sum_{\substack{i,j=1 \\ i\neq j}}^n \eta \frac{m^i m^j}{4\pi|x_\eta^i - x_\eta^j|}(1-o(1)) + c_2\beta_\eta^2, \end{aligned} \quad (3.5.6)$$

for constant $c_1, c_2 > 0$ and independent of η , and $M = \sum_{i=1}^n m_\eta^i$.

For the scale of concentration when $n \geq 2$, let $|x_\eta^k| = \beta_\eta$ and $|x_\eta^k - x_\eta^l| = \lambda_\eta$ where $k \neq l \in \{1, 2, ..., n\}$. Therefore, by matching the upper bound (3.3.1) with the lower bound (3.5.6), here exists

$$\begin{aligned} e_0(M) - M\rho_{\max} + \eta^{2/3}\mu_0 &\geq \widetilde{\mathsf{E}_\eta}(v_\eta) \\ &\geq e_0(M) - M\rho_{\max} + \eta\frac{m_k m_l}{4\pi|x_\eta^k - x_\eta^l|}(1-o(1)) + c_2\beta_\eta^2 \\ &\geq e_0(M) - M\rho_{\max} + \eta\frac{c_1}{\lambda_\eta} + c_2\beta_\eta^2. \end{aligned}$$

Therefore,

$$\lambda_\eta \geq C_1\eta^{1/3} \quad \text{and} \quad \beta_\eta \leq C_2\eta^{1/3}0$$

for some constants $C_1, C_2 > 0$.

Consider the case $n = 1$, since there are no interaction terms, there exists a simpler lower bound as

$$\mathsf{E}_\eta(v_\eta) \geq e_0(M) - M\rho_{\max}) + |y_\eta|^2 - o(\eta) = e_0(M) - M\rho_{\max} + \beta_\eta^2 - o(\eta) \tag{3.5.7}$$

Therefore, be matching the upper bound (3.3.2) with the lower bound (3.5.7), here exists

$$e_0(M) - M\rho_{\max} + \beta_\eta^2 - \mathcal{O}(\eta) \leq \mathsf{E}_\eta(v_\eta) \leq e_0(M) - M\rho_{\max} + \mathcal{O}(\eta).$$

As a result, $|y_\eta| = \beta_\eta \leq \mathcal{O}(\eta^{1/2})$.

Lemma 3.5.1. In conclusion, let $n \in \mathbb{N}$ is given as in Lemma 3.3.2 . When $n \geq 2$, points $\{x_\eta^i\}_{i=1}^n$ satisfy

$$x_\eta^i = \mathcal{O}(\eta^{1/3}), \text{ for each } i = 1, 2, ..., n.$$

When $n = 1$, points y_η from above satisfy

$$y_\eta = \mathcal{O}(\eta^{1/2}).$$

Now, everything is ready to prove the main result - Theorem 3.2.1.

Proof of Theorem 3.2.1

Let $v_\eta \in BV(\mathbb{T}^3; \{0, 1/\eta^3\})$ with $\int_{\mathbb{T}^3} v_\eta dx = M$ be a minimizer of E_η for each $\eta > 0$. Recall Lemma 3.3.2, there exists $n \in \mathbb{N}, \{m^i\}_{i=1}^n \in \mathcal{M}_1, \{x_\eta^i\}_{i=1}^n \subset \mathbb{T}^3$. Therefore, by the second part of Lemma 3.5.1 and (3.3.7) of Lemma 3.3.2, (II) holds when $n = 1$.

When $n \geq 2$, there exist $x_\eta^i = \mathcal{O}(\eta^{1/3}), i = 1, 2, ..., n$ by the first part of Lemma 3.5.1. Therefore, there exist bounded sequences $\{\eta^{-1/3} x_\eta^i\}_{\eta>0} \subset \mathbb{T}^3, i = 1, 2, ..., n$.

The precise lower bound is

$$
\begin{aligned}
\mathsf{E}_\eta(v_\eta) &\geq \sum_{i=1}^{n} \mathsf{E}_\eta(u^i_\eta) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \eta \int_{\mathbb{T}^3}\int_{\mathbb{T}^3} \frac{u^i_\eta u^j_\eta}{4\pi|x-y|} dxdy + \mathcal{O}(\eta) \\
&\geq \sum_{i=1}^{n} \left(\mathcal{G}(U^i_\eta) - \int_{U^i_\eta} \rho(x^i_\eta + \eta\xi) d\xi \right) \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \eta \int_{U^i_\eta}\int_{U^j_\eta} \frac{1}{4\pi|(x^i_\eta - x^j_\eta) + \eta(\xi - \zeta)|} d\xi d\eta + \mathcal{O}(\eta) \\
&\geq \sum_{i=1}^{n} e_0(m^i_\eta) - M\rho_{\max} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \eta \int_{U^i_\eta}\int_{U^j_\eta} \frac{1}{4\pi|(x^i_\eta - x^j_\eta) + \eta(\xi - \zeta)|} d\xi d\eta \\
&\quad + \sum_{i=1}^{n} \int_{U^i_\eta} [\rho_{\max} - \rho(x^i_\eta + \eta\xi)] d\xi + \mathcal{O}(\eta) \\
&\geq e_0(M) - M\rho_{\max} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \eta^{2/3} \int_{U^i_\eta}\int_{U^j_\eta} \frac{1}{4\pi|\eta^{-1/3}(x^i_\eta - x^j_\eta) + \eta^{2/3}(\xi - \zeta)|} d\xi d\eta \\
&\quad + \eta^{2/3} \sum_{i=1}^{n} \int_{U^i_\eta} q(\eta^{-1/3} x^i_\eta + \eta^{2/3}\xi) d\xi + o(\eta^{2/3})
\end{aligned}
$$

Thus, by Dominated Convergence,

$$
\lim_{\eta\to 0} \int_{U^i_\eta}\int_{U^j_\eta} \frac{1}{4\pi|\eta^{-1/3}(x^i_\eta - x^j_\eta) + \eta^{2/3}(\xi - \zeta)|} d\xi d\eta = \frac{m^i m^j}{4\pi|x_i - x_j|}
$$

and

$$
\lim_{\eta\to 0} \int_{U^i_\eta} q(\eta^{-1/3} x^i_\eta + \eta^{2/3}\xi) d\xi = m^i q(x_i)
$$

since each $\eta^{-1/3} x^i_\eta \to x^i$ and $U^i_\eta \to \Omega_i$ globally. Therefore, the desired lower bound is

$$
\mathsf{E}_\eta(v_\eta) \geq e_0(M) - \rho_{\max} M + \eta^{2/3} \left\{ \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{m^i m^j}{4\pi|x_i - x_j|} + \sum_{i=1}^{n} m^i q(x_i) \right\} + o(\eta^{2/3}).
$$

□

4 The Liquid Drop Model with Background Potential

4.1 Introduction

The liquid drop model predicts the spherical shape of small nuclei and the non-existence of arbitrarily large nuclei. However, the competition between the surface tension and Coulombic repulsion makes these predictions not accurate.

Consider the variant model as

$$e_Z(M) := \inf\{\mathsf{E}_Z(\Omega) : \Omega \in \mathbb{R}^d, |\Omega| = M\}, \tag{4.1.1}$$

where the energy functional E_Z is

$$\mathsf{E}_Z(\Omega) := \mathrm{Per}(\Omega) + \int_\Omega \int_\Omega \frac{1}{|x-y|^s} dxdy - Z \int_\Omega \frac{1}{|x|^p} dx \tag{4.1.2}$$

with $0 < p < s < d$ and $d \geq 2$. $\mathrm{Per}(\Omega)$ is the perimeter of the set Ω in the sense of Caccioppoli and it is defined as

$$\mathrm{Per}(\Omega) = \sup\Big\{ \int_\Omega \mathrm{div}\ \phi dx : \phi \in C_0^1(\mathbb{R}^d; \mathbb{R}^d), ||\phi||_{L^\infty(\mathbb{R}^d)} \leq 1 \Big\}.$$

The small Z regime from Gamow's liquid drop model [3] models the shape of an atomic nucleus. Gamow' model is equivalent to the variant model (4.1.1) with $d = 3, s = 1$, and $Z = 0$ as

$$\text{minimize } \mathrm{Per}(\Omega) + \int_\Omega \int_\Omega \frac{1}{|x-y|} dxdy \text{ over } \Omega \subset \mathbb{R}^3 \text{ with } |\Omega| = M. \tag{4.1.3}$$

Note that the non-existence of minimizers for large M is associated with the breakup of droplets tending to infinity. In physics, for a large M, there is the expectation of other forces to be involved to restore the existence. Then, it is possible to predict a structured configuration of droplets. Thus, add an external attractive potential

$$V(x) = -\frac{Z}{|x|^p} \tag{4.1.4}$$

to (4.1.3) for $Z > 0$ and $0 < p \leq 1$ is a way to introduce the effect. This attractive is the background nucleus. Set this background nucleus as the center at the origin. They have slower decay than the Coulombic nonlocal interaction term for longer range. From [26] and [27], the effect of V increases the critical threshold in M for the non-existence of minimizers is proved at the case of Coulombic attraction, which is the physical case of $p = 1$. In addition, when $p < 1$, the existence is restored for all M [28]. Therefore, it is possible to think of the attractive long-range potential as regularizing the generalized liquid drop model. Thus, the next step is to focus on the structure of minimizers in small Z regime. The following results are particular configurations of generalized minimizers [[10], Definition 1.1] of the the liquid drop model.

4.2 Main Results

Theorem 4.2.1. For all $Z > 0$ and any $M > 0$, the minimum $e_z(M)$ is attained.

This result is a generalization result in [28]. This result confirms that the presence of the external potential (4.1.4) with $p < s$ indeed restores existence for all masses $M > 0$. The continued research is in minimizers of E_Z in the limit of $Z \to 0$. For $d \geq 2$, there exists $m^* = m^*(d, s) > 0$ such that $Z = 0$ as

$$e_0(M) := \inf\{\mathsf{E}_0(\Omega) : \Omega \subset \mathbb{R}^d, |\Omega| = M\}, \tag{4.2.1}$$

which does not admit minimizers for $M > m^*$ and $s \in (0, 2)$. Therefore, a sequence of minimizers Ω_Z of the functional E_Z lose compactness as $Z \to 0$ when $M > m^*$. For small $Z > 0$, Ω_Z is composed of a finite number of disjoint compact components that are separated by a distance on the order of $Z^{1/(s-p)}$ [19]. Besides, after rescaling by $Z^{1/(s-p)}$, a discrete interaction energy is optimized by the way of how the components are arranged

$$\mathsf{F}_{N,\underline{m}}(y_0, y_1, ..., y_N) := \sum_{\substack{i,j=0\\ i\neq j}}^{N} \frac{m^i m^j}{|y_i - y_j|^s} - \sum_{i=1}^{N} \frac{m^i}{|y_i|^p}, \tag{4.2.2}$$

where $\underline{m} = (m^0, m^1, ..., m^N)$ with $\sum_{i=0}^{N} m^i = M$, and

$$\sum_{N} := \{(y_0, y_1, ..., y_N) \subset \mathbb{R}^{3(d+1)} : y_0 = 0\}. \tag{4.2.3}$$

Theorem 4.2.2. The following is the most important main result that describes the structure of minimizers of E_Z for small $Z > 0$ [19].

Let Ω_Z be minimizers of E_Z for $Z > 0$. Thus, there exists a subsequence $Z_n \to 0$ for any sequence $Z \to 0$ such that either

(I) there exists a set E^0 with $|E^0| = M$ which minimizes $e_0(M)$, for which $\Omega_{Z_n} \to E^0$ globally as

$$\chi_{\Omega_{Z_n}} \to \chi_{E^0} \in L^1(\mathbb{R}^d) \text{ as } n \to \infty;$$

or

(II) there exist:

- (i) $N \in \mathbb{N}$;
- (ii) $(m^0, m^1, ..., m^N), m^i > 0$ with $\sum_{i=0}^{N} m^i = M$;
- (iii) $x_n^0, x_n^1, ..., x_n^N \in \mathbb{R}^d$ with $x_n^0 = 0$, and $|x_n^i| \to \infty$ for $i \neq 0$, and $|x_n^i - x_n^j| \to \infty$ for $i \neq j$ as $n \to \infty$;
- (iv) $E^0, E^1, ..., E^N$ compact sets of finite perimeter with $|E^i| = m^i \neq 0$ for $i = 0, 1, ..., N$;

such that $\Omega_n := \Omega_{Z_n}$ satisfies the following:

$\partial^*\Omega_n \in C^{1,1/2}$ and for fixed $R > 0$ such that all $E^i \subset B_R(0)$,

$$(\partial^*\Omega_n - x_n^i) \cap B_R(0) \to \partial^* E^i \text{ in } C^{1,\alpha} \text{ for all } \alpha \in (0, \frac{1}{2}); \tag{4.2.4}$$

$$\left| \Omega_n \triangle \left[E^0 \cup \bigcup_{i=1}^{N} (E^i + x_n^i) \right] \right| \to 0; \tag{4.2.5}$$

$$E^i \text{ attains the minimum in (4.2.1) as } e_0(m^i) = \mathsf{E}_0(E^i), i = 0, 1, ..., N; \tag{4.2.6}$$

$$Z_n^{\frac{1}{s-p}} x_n^i \to y_i \text{ as } n \to \infty, i = 1, 2, ..., N, \text{ where } (0, y1, ..., y_N) \text{ minimize } \mathsf{F}_{N,\underline{m}} \text{ over } \sum_N. \tag{4.2.7}$$

For Gamow's model, the minimizers are only for small mass M and connected. However, for $Z > 0$, the minimizers of E_Z are always for any M but not connected for mass $M > m^*$.

Definition 4.2.1. Then, adapt the definition of E_Z from [[10], Definition 4.3] to conclude the resulting structure (4.2.5) and (4.2.6). Let $Z \geq 0$ and $M > 0$. The generalized minimizer of E_Z is a finite collection $(E^0, E^1, ..., E^N)$ of sets of finite perimeter such that

(i) $|E^i| := m^i, i = 0, 1, ..., N$, with $\sum_{i=0}^{N} m^i = M$;

(ii) E^0 attains the minimum in $e_Z(m^0)$ and E^i attains $e_0(m^i), i = 1, 2, ..., N$;

(iii) $e_Z(M) = e_Z(m^0) + \sum_{i=1}^{n} e_0(m^i)$.

Therefore, the result is improved from the result in [10] as they proved the existence of generalized minimizers for the Gamow problem $Z = 0$. The improvement is that the result follows immediately from the concentration characterized up to sets of vanishingly small measure, and along subsequences by a generalized minimzers.

Corollary 4.2.1. Let $Z \geq 0$, $M > 0$ and $\{\Omega_n\_n \in \mathbb{N}\}$ be any minimizing sequence for $e_Z(m)$. Thus, there exists a subsequence, $\mathbb{N} \geq 0$, and a generalized minimizer $(E^0, E^1, ..., E^N)$ of E_Z, with

$$\left|\Omega_n \triangle \left[E^0 \cup \bigcup_{i=1}^{N}(E^i + x_n^i)\right]\right| \to 0$$

for a sequnece of translations $(x_n^i)_{n\in\mathbb{N}}^{i=1,2,...,N}$.

The above Theorem 4.2.2 shows that the family Ω_Z of minimizers of E_Z makes a particular selection of a generalized minimizer for the generalized liquid drop problem E_0. Besides, the special choice of generalized minimizer in this way may not be canonical such that the sets and the pattern they form as $Z \to 0$ depend on the choice of external potential [19].

Furthermore, in [5], Bonacini and Cristoferi show that there exists a critical value $\overline{s}(d)$ of the power in the Riesz kernel such that if $s \in (0, \overline{s}(d))$. Thus, the minimizers of $e_0(M)$ must be balls. It means that for small s, the critical mass for existence exactly coincides with the critical value at which minimizers must be balls [19]. As a result, the following theorem describes minimizers for small $Z > 0$ as a finite configuration of balls of equal radius.

Theorem 4.2.3. Assume $0 < s < \overline{s}(d)$ and $0 < p < s < d$. Then, the sets E^i in Theorem 4.2.2 are all balls with equal volume $m^i = M/(N+1), i = 0, 1, ..., N$.

4.3 Concentration-Compactness Structure

Furthermore, this section focuses on the concentration-compactness structure of minimizing sequences for E_Z. There are some studies in this area from [23] or Chapter 29 of [29]. Here, the result from [6] by Frank and Lieb is used as it is the well-suited one.

Consider a sequence of sets $E_n \to E$ is globally in $\mathbb{R}^d$, while the measure of the symmetric difference $|E_n \triangle E| \to 0$. Thus, similarly, $E_n \to E$ is locally for every compact $K \subset \mathbb{R}^d$, $(K \cap E_n) \to (K \cap E)$ globally. Therefore, when the local convergence is only L^1_{loc} convergence of the characteristic functions, the global convergence is equivalent to convergence of the characteristic functions $\chi_{E_n} \to \chi_E$ in $L^1(\mathbb{R}^d)$.

Lemma 4.3.1. Let $Z \in [0, \infty)$ be fixed and $\{\Omega_n\}_{n\in\mathbb{N}}$ a minimizing sequence for $e_Z(M)$. Thus, there exists a subseqence such that either

(A) there exists a set E^0 with $|E^0| = M$ that minimizes $e_Z(M)$, for which $\Omega_n \to E^0$ globally such that $\chi_{\Omega_n} \to \chi_{E^0}$ in $L^1(\mathbb{R}^d)$ as $n \to \infty$; or

(B) there exist $N \in \mathbb{N}$; $\{x_n^1, x_n^2, ..., x_n^N\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ with $|x_n^i| \to \infty$ and sets of finite perimeter $\{F_n^0, F_n^1, ..., F_n^N, \Omega_n^N\}_{n\in\mathbb{N}}$ such that $|x_n^j - x_n^j| \to \infty, i \neq j$; with

$$\Omega_n = F_n^0 \cup \left[\bigcup_{i=1}^{N}(F_n^i + x_n^i)\right] \cup \Omega_n^N, \tag{4.3.1}$$

a disjoint union with components satisfying the following:

(i) $\Omega_n^N \to \emptyset$ and $F_n^i \to E^i$, globally in $\mathbb{R}^d$, with $m^i := |E^i| > 0$ for all $i = 1, 2, ..., N$ and $|E^0| > 0$ for $Z > 0$;

(ii) $M = \sum_{i=0}^N |E^i| = \lim_{n\to infty}(\sum_{i=0}^N |F_n^i + |\Omega_n^N|)$;

(iii) E^i attain the minimum for $e_0(m^i)$ for each $i = 1, 2, ..., N$;

(iv) E^0 attains the minimum for $e_Z(m^0)$;

(v) $e_Z(M) \geq e_Z(m^0) + \sum_{i=1}^N e_0(m^i)$.

Note that the collection of sets $\{E^0, ..., E^n\}_{n\in\mathbb{N}}$ is a generalized minimizer of E_Z for any $Z \geq 0$. In [10], the existence of generalized minimizers for the case $Z = 0$ is proved. The truncation of energy E_0 and obtain density bounds for minimizers of the truncated energy have been used. Here, to prove this lemma, a better way is obtained by [19] with qualitative information of the structure of minimizing sequences. The following lemma is needed to deal with the confinement term.

Lemma 4.3.2. Assume A_n is a sequence of measurable sets with $|A_n| = M$ and $A_n \to 0$

locally. It means that $|A_n \cap K| \to 0$ for any compact K. Thus,

$$\lim_{n\to\infty} \int_{A_n} \frac{1}{|x|^p} dx = 0.$$

There are several steps to prove this lemma. Please see [[19], Section 2] for the detailed proof.

4.4 Regularity Results

Next, in this section, the research is mainly focused on the limiting finite-dimensional energy $\mathsf{F}_{N,\underline{m}}(y_0, y_1, ..., y_N)$. It is not clear whether this minimizing sequences for this energy with some number of points diverging to infinity split or not [19]. However, the following proposition will show that $\mathsf{F}_{N,\underline{m}}$ attains its minimizer for all choices of N and $\underline{m}$.

In (4.2.2), there exists a minimizer of the finite-dimensional energy functional $\mathsf{F}_{N,\underline{m}}$. Then, define

$$\mu_{N,\underline{m}} := \inf_{\sum_N} \mathsf{F}_{N,\underline{m}}.$$

Then, consider any minimizing sequence $\{x_n^i\}_{n\in\mathbb{N}}, i = 1, 2, ..., N$ in $\sum_N$. There exists $\mu_{N,\underline{m}} = \lim_{n\to\infty} \mathsf{F}_{N,\underline{m}}(0, x_1^n, ..., x_N^n)$. If all the sequences $\{x_n^i\}_{n\in\mathbb{N}}$ are bounded, then they are convergence to a minimizer along some subsequence. Otherwise, assume there is an integer $k \in (0, 1, ..., N-1)$ and a subsequence that

$$\begin{cases} x_n^i \overset{n\to\infty}{\longrightarrow} a^i, & \forall i = 0, 1, ..., k \\ |x_n^i| \overset{n\to\infty}{\longrightarrow} \infty, & \forall i = (k+1), ..., N. \end{cases} \tag{4.4.1}$$

For the case $k \geq 1$, decompose $\mathsf{F}_{N,\underline{m}}$ into

$$\mathsf{F}_{N,\underline{m}}(0, x_n^1, ..., x_n^N) = \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, x_n^1, ..., x_n^k) + \mathsf{F}_{N-k,(m^{k+1},m^{k+2},...,m^N)}(x_n^{k+1}, ..., x_n^N) + I_{k,N}, \tag{4.4.2}$$

where

$$I_{k,N} = \sum_{i=0}^{k} \sum_{j=k+1}^{N} \frac{m^i m^j}{|x_n^i - x_n^j|^s}.$$

Therefore, by using (4.4.1), there exists

$$\begin{aligned} \mu_{N,\underline{m}} &\geq \liminf_{n\to\infty} \left[\mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, x_n^1, ..., x_n^k) + \sum_{\substack{i,j=k+1 \\ i\neq j}}^{N} \frac{m^i m^j}{|x_n^i - x_n^j|^s} \right] \\ &\geq \liminf_{n\to\infty} \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, x_n^1, ..., x_n^k) \\ &= \liminf_{n\to\infty} \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, a_1, ..., a_k). \end{aligned} \tag{4.4.3}$$

In addition, define a new configuration given by the points $\{a_1, ..., a_k, Ry_1, ..., Ry_{N-k}\}$ with $\{y_1, ..., y_{N-k}\}$ distinct points on the unit sphere $|y_j| = 1$, and $R > 0$. By the same decomposition in (4.4.2), there exist

$$\begin{aligned} \mathsf{F}_{N,\underline{m}}(0, a_1, ..., a_k, Ry_1, ..., Ry_{N-k}) &= \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, a_1, ..., a_k) \\ &\quad + \mathsf{F}_{N-k,(m^{k+1},m^{k+2},...,m^N)}(Ry_1, ..., Ry_{N-k}) + \tilde{I}_{k,N}. \end{aligned} \tag{4.4.4}$$

$\tilde{I}_{k,N}$ represents the interaction terms. If $|a_i| < R_0, i = 1, 2, ..., k$ for some $R_0 > 0$ and $R > 2R_0$, the interaction terms may be estimated by

$$\tilde{I}_{k,N} \leq C_1(k, N, \underline{m}) R^{-s}.$$

Moreover, for some constant $C_2 > 0$, there exists

$$\mathsf{F}_{N-k,(m^{k+1},m^{k+2},...,m^N)}(Ry_1, ..., Ry_{N-k}) \leq \sum_{\substack{i,j=1 \\ i\neq j}}^{N-k} \frac{m^{k+i} m^{k+j}}{|Ry_i - Ry_j|^s} \leq C_3(k, N, \underline{m}) R^{-s}$$

since $|Ry_i - Ry_j| \geq C_2 R, i \neq j$. On the other hand,

$$\sum_{i=1}^{N-k} m^{k+i} |Ry_i|^{-p} = R^{-p} \sum_{i=1}^{N-k} m^{k+i} \geq C_4(k, N, \underline{m}) R^{-p}.$$

Thus, by (4.4.4) and some large enough $R > R_0 > 0$, there exits

$$\begin{aligned} \mu_{N,\underline{m}} &\leq \mathsf{F}_{N,\underline{m}}(0, a_1, ..., a_k, Ry_1, ..., Ry_{N-k}) \\ &\leq \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, a_1, ..., a_k) - C_4(k, N, \underline{m}) R^{-p} + O(R^{-s}) \\ &< \mathsf{F}_{k,(m^0,m^1,...,m^k)}(0, a_1, ..., a_k) \end{aligned} \tag{4.4.5}$$

For the case $k = 0$, if $|x_n^i| \to \infty$ for each $i = 1, 2, ..., N$, there exists

$$\mu_{N,\underline{m}} \geq \liminf_{n\to\infty} \sum_{\substack{i,j=0 \\ i\neq j}}^{N} \frac{m^i m^j}{|x_n^i - x_n^j|^s} \geq 0.$$

Thus, the same construction is produced as in (4.4.5) yields the contradictory estimate $\mu_{N,\underline{m}} < 0$. Therefore, the entire minimizing sequence must remain bounded.

Proposition 4.4.1. As a result, for any $N \in \mathbb{N}$ and $\underline{m}$, the functional $\mathsf{F}_{N,\underline{m}}$ attains its minimum $\mu_{N,\underline{m}} < 0$ on the admissible class $\sum_N$.

Next, consider the infimum of the regularized energies E_Z converges to the infimum of E_0. Let Ω_Z be a minimizer of E_Z for any $Z > 0$ and $M > 0$. Thus, $e_Z(M) \leq e_0(M)$ for all $Z > 0$. Besides, $\omega_d = |B_1(0)|$ represents the the volume of the unit ball in $\mathbb{R}^d$ such that

$$\begin{aligned} \mathsf{E}_0(\Omega_Z) &= \mathsf{E}_Z(\Omega_Z) + Z \int_{\Omega_Z} \frac{1}{|x|^p} dx \\ &\leq \mathsf{E}_Z(\Omega_Z) + Z \int_{B_1(0)} \frac{1}{|x|^p} dx + Z|\Omega_Z \cap (\mathbb{R}^d \backslash B_1(0))| \\ &\leq \mathsf{E}_Z(\Omega_Z) + \big(\frac{\omega_d}{d-p} + M\big) Z. \end{aligned}$$

Lemma 4.4.1. As a result, $\lim_{Z\to 0} e_Z(M) = e_0(M)$.

Furthermore, the following is the key to obtain regularity properties for a family of minimizers of the functional E_Z.

First, consider the constraint $|\Omega_Z| = M$, by [[30], Section 2] and [[5], Theorem 2.7], it may be replaced by a penalization. For $\lambda > 0$, define the penalized functional

$$\mathcal{F}_Z^\lambda(F) := \mathsf{E}_Z(F) + \lambda||F| - |\Omega_Z|| = \mathsf{E}_Z(F) + \lambda||F| - M|.$$

In addition, the unconstrained minimizer of $\mathcal{F}_Z^\lambda$ coincides with the mass-constrained minimizer of E_Z. Thus, λ may be chose independently of Z since the existence of a constant $\lambda = \lambda_Z > 0$ for each fixed $Z >$) satisfies the claim in a minor modification of [[5], Theorem 2.7]. If there does not exist such λ, then there exist sequences $Z_n \to 0, \lambda_n \to 0$, and sets $E_n \subset \mathbb{R}^d, |E_n| \neq M$, with $\mathcal{F}_{Z_n}^{\lambda_n}(E_n) < \mathcal{F}_{Z_n}^{\lambda_n}(\Omega_{Z_n})$. Thus, $|E_n| \to M$ since $\lambda_n \to \infty$

Furthermore, define sets $\tilde{E}_n = t_n E_n, t_n = [M/|E_n|]^{1/d}$. Thus, $|\tilde{E}_n| = M$. Therefore, by scaling, there exists

$$\begin{aligned}
\mathcal{F}_{Z_n}^{\lambda_n}(\tilde{E}_n) = \mathsf{E}_{Z_n}(\tilde{E}_n) &= t_n^{d-1}\mathrm{Per}(E_n) + t_n^{2d-s}\mathcal{D}(E_n, E_n) - t_n^{d-p} Z_n \int_{E_n} |x|^{-p} dx \\
&= \mathcal{F}_{Z_n}^{\lambda_n}(E_n) + (t_n^{d-1} - 1)\mathrm{Per}(E_n) + (t_n^{2d-s} - 1)\mathcal{D}(E_n, E_n) \\
&\quad - (t_n^{d-p} - 1) Z_n \int_{E_n} |x|^{-p} dx - \lambda_n |t_n^{d-1} - 1||E_n| \\
&\leq \mathcal{F}_{Z_n}^{\lambda_n}(E_n) + |t_n^{d-1} - 1||E_n| \left[\mathsf{E}_0(E_n) \frac{(t_n^{d-1} + t^{2d-s} - 2)}{|t_n^{d-1} - 1||E_n|} - \lambda_n \right] \\
&< \mathcal{F}_{Z_n}^{\lambda_n}(E_n),
\end{aligned}$$

as $\lambda_n \to \infty$ since the term in brackets is negative. Therefore, this contradicts the definition of E_n as minimizers of $\mathcal{F}_{Z_n}^{\lambda_n}$. As a result, there exist $\lambda > 0$ for all $0 < Z \leq 1$,

$$\min \mathcal{F}_Z^\lambda = \mathcal{F}_Z^\lambda(\Omega_Z) = \mathsf{E}_Z(\Omega_Z). \tag{4.4.6}$$

Next, define

$$\mathcal{V}(F) := \int_F \frac{1}{|x|^p} dx,$$

for any fixed $r > 0$, and assuming $B_r(x_0) \cap B_\delta(0) = \emptyset$ and $F \subset \mathbb{R}^d$ with $\Omega_Z \triangle F \subset B_r(x_0)$. Then, $\mathsf{E}_Z(\Omega_Z) = \mathcal{F}_Z^\lambda(\Omega_Z) \leq \mathcal{F}_Z^\lambda(F)$ implies that

$$\begin{aligned}\mathrm{Per}(\Omega_Z) &\leq \mathrm{Per}(F) + (\mathcal{D}(F,F) - \mathcal{D}(\Omega_Z,\Omega_Z)) + (\mathcal{V}(\Omega_Z) - \mathcal{V}(F)) + \lambda||F| - M| \\ &\mathrm{Per}(F) + (C_0 + \delta^{-p} + \lambda)|\Omega_Z \triangle F|.\end{aligned}$$

The difference of the nonlocal terms is estimated in [[5], Proposition 2.3]. For estimating the confinement term, use $|x|^-p \in L^\infty(\mathbb{R}^d \backslash B_\delta(0))$. Thus, Ω_Z are (ω, r)-minimizers of the perimeter functional in $\mathbb{R}^d \backslash B_\delta(0)$ with $\omega = C_0 + \delta^{-p} + \lambda$ and any $r > 0$.

Lemma 4.4.2. As a result, the family of minimizers $\{\Omega_Z\}_{Z\in(0,1]}$ of E_Z is (ω, r)-minimizers of the perimeter functional in $\mathcal{O} := \mathbb{R}^d \backslash B_\delta(0)$ for any $\delta > 0$, with $\omega, r > 0$ uniformly chosen for $Z \in (0, 1]$. For all $F \subset \mathbb{R}^d$ with $\Omega_Z \triangle F \subset B_r(x_0) \subset \mathbb{R}^d \backslash B_\delta(0)$, there exists

$$\mathrm{Per}(\Omega_Z) \leq \mathrm{Per}(F) + \omega|\Omega_Z \triangle F|.$$

Lemma 4.4.3. Moreover, the following lemma concludes the regularity results for (ω, r)-minimizers. From Theorem 21.8, Theorem 21.14 and Theorem 26.6 in [29], let $\mathcal{O} \subset \mathbb{R}^d$ be an open set:

(i) If $E \subset \mathbb{R}^d$ is an (ω, r)-minimizer of perimeter in $\mathcal{O}$, then $\partial^* E \cap \mathcal{O}$ is a $C^{1,\alpha}$ hypersurface for any $\alpha \in (0, 1/2)$.

(ii) If $E_n \subset \mathbb{R}^d$ is a sequence of uniformly (ω, r)-minimizers of perimeter in $\mathcal{O}$ with $E_n \to E_\infty$ locally in $\mathcal{O}$, then there exists $x_\infty \in \partial E_\infty$ for any any sequence $x_n \in \partial E_n$ with $x_n \to x_\infty$. In addition, if $x_n \in \partial^* E_n$, then $x_\infty \in \partial^* E_\infty$ and the normal vectors satisfy $\nu(x_n) \to \nu(x_\infty)$.

4.5 Proof

Proof of Theorem 4.2.1

The proof of Theorem 4.2.1 is given in [28] for the Newtonian case as $s = 1$ with more general confinement terms.

Proof of Theorem 4.2.2

The idea to prove the Theorem 4.2.2 is to use the the concentration-compactness lemma to minimize sequences of E_Z. In addition, an expansion of the energy E_Z is required up to the third-order term in Z. Thus, to establish it, there is one approach is similar to the previous research of the concentration of droplets in a sharp interface model of diblock copolymers under confinement. Therefore, combine the compactness of a sequence of minimizers Ω_Z with regularity results stemming from the classical regularity properties of the perimeter functional improving the error estimates in [6].

To prove the Theorem 4.2.2, there are some steps to follow by using the regularity of minimizing sets to improve the precision of the lower bound defined in the previous lemmas. Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be a sequence of minimizers for $e_{Z_n}$where $\Omega_n := \Omega_{Z_n}$ is the sequence of minimizers for e_{Z_n} with $Z_n \to 0$. From Lemma 4.4.1, $\{\Omega_n\}$ form a minimizing sequence for e_0. From Lemma 4.3.1, there exist either (A) or assertions (i), (ii), and (4.2.5), (4.2.6), in (iii) of (B) in Theorem 4.2.2. In addition, from Lemma 4.4.2 and Lemma 4.4.3, (4.2.4) is followed.

By using the uniform (ω, r)-minimality to show that

$$\Omega_n = F_n^0 \cup \left[\bigcup_{i=1}^{N} (F_n^i + x_n^i) \right],$$

splits with no error in the perimeter, with the remainder set $\Omega_n^N = \emptyset$. Besides, define

$$\tilde{F}_n^i = F_n^i + x_n^i \quad \text{and} \quad \hat{\Omega}_n^i = \Omega_n^{i-1} - x_n^i, \quad i = 0, 1, ..., N.$$

There are essentially bounded domains with smooth $\partial^* E^i$, E^i, the minimzers of $e_0(m^i)$ [[9], Proposition 2.1 and Lemma 4.1]. Thus, each $E^i \subset B_{R/2}(0)$ for each $i = 0, 1, ..., N$ and $R > 0$. It is possible to choose the radii r_n in [[6], Lemma 2.2] such that $r_n \in (R, 2R)$ when defining the $F_n^i = \hat{\Omega}_n^{i-1} \cap B_{r_n}(0)$ since E^i is bounded. As $\hat{\Omega}_n^i \to E^i$ locally, it converges globally in $\mathcal{O} := B^{2R}(0)$ since Lemma 4.4.2 ensures $\hat{\Omega}_n^i$ is a family of uniformly (ω, r)-minimizers in $\mathcal{O}$. By the regularity result (part (ii) of Lemma 4.4.3), for all sufficiently large n, $\hat{\Omega}_n^i \cap \mathcal{O} \to E^i \subset B_{R/2}(0)$ in Hausdorff norm, and particularly $\hat{\Omega}_n^i \cap B_{2R}(0) \subset B_R(0)$. Moreover, when $i = 0$, define the open set $\tilde{\mathcal{O}} := B_{2R}(0) \backslash \overline{B_\delta(0)}$ for any $\delta \in (0, R\backslash 2)$. Therefore, Ω_n are uniformly (ω, r)-minimizers in $\hat{\mathcal{O}}$ since Ω_n are not necessarily (ω, r)-minimizers in a neighborhood of 0 when $i = 0$. As a result, $\Omega_n \cap [B_{2R}(0) \backslash B_R(0)] = \emptyset$ for all sufficiently large n.

Furthermore, assume $\Omega_n^N \neq \emptyset$ for all $n \in \mathbb{N}$. Then $\Omega_n^N \to$ globally since $|\Omega_n^N| \to 0$ by Lemma 4.3.1. There exist $y_n \in \partial \Omega_n^N$ for each n since Ω_n is the (ω, r)-minimizing sequence each $\partial^* \Omega_n^N$ is a smooth hypersurface. Besides, by (ii) of Lemma 4.4.3, there is a contradiction as 0 liew on the boundary of the limit set of the $\hat{\Omega}_n^N$. As a result, there exists $\Omega_n^N = \emptyset$ for large n. As $|x_n^i - x_n^j| \to \infty$ for $i \neq j$ and $G_n^i \cap B_R(0) = \emptyset$, the components are well separated such that for each sufficiently large n, there exists

$$\text{Per}(\Omega_n) = \sum_{i=0}^{N} \text{Per}(F_n^i). \tag{4.5.1}$$

Besides,

$$M = \sum_{i=1}^{N} m^i = \sum_{i=1}^{N} m_n^i \tag{4.5.2}$$

holds for all sufficiently large n as (4.5.1) implies the equality of masses before and after passing to the limit.

Next, it is possible to choose $k \in \{1, 2, ..., N\}$ and a subsequence along $|x_n^k| =\in \{|x_n^j| : j = 1, 2, ..., N\}$ since there are only finitely many components. Thus, consider sets $\breve{\Omega}_n := \Omega_n - x_n^k$. The modification only affects the confinement term $\mathcal{V}$ since the perimeter and nonlocal terms in E_Z are translation invariant. Thus, there exists a disjoint decomposition

$$\breve{\Omega}_n = F_n^0 \cup F_n^k \cup \left[\bigcup_{\substack{i=1 \\ i \neq k}}^{N} (F_n^i + y_n^i) \right] \cup \Omega_n^N,$$

where $y_n^i = x_n^i - x_n^k$ with $|y_n^i| \to \infty, i \neq k$. Thus, for all $j = 1, 2, ..., N$ and all $i \neq k$, when $\mathcal{V}(F_n^k) \to \mathcal{V}(E^k) > 0$, there exists $\mathcal{V}(F_n^j + x_n^j) \to 0$ and $\mathcal{V}(F_n^i + y_n^i) \to 0$. Therefore, there exist the contradictions of the minimality of Ω_n such that

$$\begin{aligned} \mathsf{E}_Z(\breve{\Omega}_n) - \mathsf{E}_Z(\Omega_n) &= -Z_n \mathcal{V}(F_n^k) - Z_n \sum_{\substack{i=1 \\ i \neq k}}^{N} \mathcal{V}(F_n^i + y_n^i) + Z_n \sum_{i=1}^{N} \mathcal{V}(F_n^i + x_n^i) \\ &= -Z_n \mathcal{V}(E^k) + o(Z_n) < 0. \end{aligned}$$

As a result, there exists $|E^0| \neq 0$.

Furthermore, to have a more refined lower bound is considerable. There exists $R > 0$ for which $F_n^i \subset B_R(0)$ for each $n \in \mathbb{N}, i = 0, 1, ..., N$. It is possible to decompose the nonlocal term since $\bigcup_{i=0}^{N} (F_n^i + x_n^i) \subset \Omega_n$ such that

$$\mathcal{D}(\Omega_n, \Omega_n) \geq \sum_{i,j=0}^{N} \mathcal{D}(\tilde{F}_n^i, \tilde{F}_n^j).$$

Define

$$R_{n,ij} := |x_n^i - x_n^j| \quad \text{and} \quad R_{n,i0} := |x_n^i|.$$

Thus, there exists

$$|x - y| \geq R_{n,ij} - 2R \geq \frac{1}{2} R_{n,ij}$$

for all $x \in \tilde{F}_n^i$, $y \in \tilde{F}_n^j$, and sufficiently large n. Then, by the mean value theorem for $f(t) = t^s$, there exists

$$\begin{aligned} ||x_n^i - x_n^j|^s - |x-y|^s| &\le s\big(\frac{1}{2}R_{n,ij}\big)^{s-1}|x_n^i - x_n^j - x + y| \\ &\le CR_{n,ij}^{s-1}(|x_n^i - x| + |x_n^j - y|) \\ &\le 2CRR_{n,ij}^{s-1}. \end{aligned}$$

Thus,

$$\left|\frac{1}{|x-y|^s} - \frac{1}{|x_n^i - x_n^j|^s}\right| = \frac{||x_n^i - x_n^j|^s - |x-y|^s|}{|x-y|^s|x_n^i - x_n^j|^s} \le \frac{C}{R_{n,ij}^{s+1}}$$

for all sufficiently large n and all $0 < s < d$. Therefore, the off-diagonal terms in the nonlocal energy can be estimated by

$$\left|\mathcal{D}(\tilde{F}_n^i, \tilde{F}_n^j) - \frac{m_n^i m_n^j}{|x_n^i - x_n^j|^s}\right| \le \int_{\tilde{F}_n^i}\int_{\tilde{F}_n^j}\left|\frac{1}{|x-y|^s} - \frac{1}{|x_n^i - x_n^j|^s}\right| dxdy \le CR_{n,ij}^{-s-1}, \tag{4.5.3}$$

with a constant C independent of n. In addition, for the confinement term, it can be evaluated in the similar way as

$$||x_n^i|^{-p} - |x|^{-p}| \le \sup_{\xi \in \tilde{F}_n^i} p|\xi|^{-p-1}|x - x_n^i| \le C|x_n^i|^{-p-1} \le CR_{n,i0}^{-p-1}.$$

Therefore,

$$\left|\int_{\tilde{F}_n^i}\frac{1}{|x|^p}dx - \frac{m_n^i}{|x_n^i|^p}\right| \le CR_{n,i0}^{-p-1}. \tag{4.5.4}$$

As a result, by combining the previous estimates and the perimeter splitting, the lower bound is

$$\begin{aligned}
\mathsf{E}_{Z_n}(\Omega_n) \geq& \sum_{i=0}^{N} \mathsf{E}_0(F_n^i) - Z_n \mathcal{V}(F_n^0) + \sum_{\substack{i,j=0\\ i\neq j}}^{N} \frac{m_n^i m_n^j}{|x_n^i - x_n^j|^s}(1 - O(R_{n,ij}^{-1})) \\
&- Z_n \sum_{i=1}^{N} \frac{m_n^i}{|x_n^i|^p}(1 + O(R_{n,i0}^{-1})) \\
\geq& \sum_{i=0}^{N} e_0(m_n^i) - Z_n \mathcal{V}(F_n^0) + \sum_{\substack{i,j=0\\ i\neq j}}^{N} \frac{m_n^i m_n^j}{|x_n^i - x_n^j|^s}(1 - O(R_{n,ij}^{-1})) \\
&- Z_n \sum_{i=1}^{N} \frac{m_n^i}{|x_n^i|^p}(1 + O(R_{n,i0}^{-1})) \\
\geq& \sum_{i=0}^{N} e_0(m_n^i) - Z_n \mathcal{V}(F_n^0) + \sum_{\substack{i,j=0\\ i\neq j}}^{N} \frac{m_n^i m_n^j}{|x_n^i - x_n^j|^s}(1 - o(1)) \\
&- Z_n \sum_{i=1}^{N} \frac{m_n^i}{|x_n^i|^p}(1 + o(1)) \text{ (by the convergence } m_n^i \to m^i).
\end{aligned} \tag{4.5.5}$$

Moreover, to have a more refined upper bound is considerable. Suppose $\Omega_t = F_n^0 \cup [\bigcup_{i=1}^{N}(F_n^i + ta^i)]$ with sets F_n^i as in Lemma 4.3.1 with points $\{a^i\}_{i=1,\dots,N} \subset \mathbb{R}^d$ with $0 < |a^i| \leq 1$. Then, substitute Ω_t into E_Z to get an upper bound.

$$\begin{aligned}
e_{Z_n}(M) \leq& \mathsf{E}_{Z_n}(\Omega_t) \\
\leq& \sum_{i=0}^{n} e_0(m_n^i) - Z_n \mathcal{V}(F_n^0) + \sum_{\substack{i,j=0\\ i\neq j}}^{N} \int_{F_n^i + ta^i} \int_{F_n^j + ta^j} \frac{1}{|x-y|^s} dxdy \\
&- Z_n \sum_{i=1}^{N} \int_{F_n^i + ta^i} |x|^{-p} dx.
\end{aligned}$$

Besides, with the same estimates as in (4.5.3) and (4.5.4), the following holds

$$\left| \mathcal{D}(F_n^i, F_n^j) - \frac{m^i m^j}{t^s |a^i - a^j|^s} \right| \leq C t^{-s-1}, \quad \left| \int_{\tilde{F}_n^i} \frac{1}{|x|^p} dx - \frac{m^i}{t^p |a^i|^p} \right| \leq C t^{-p-1},$$

for constant C in dependent of t. The following upper bound of the form holds by choosing

$t = t_n := Z_n^{-1/(s-p)}$ as

$$e_{Z_n}(M) \leq \mathsf{E}_{Z_n}(\Omega_{t_n}) \leq \sum_{i=0}^{N} e_0(m_n^i) - Z_n \mathcal{V}(F_n^0) + Z_n^{s/(s-p)} \mathsf{F}_{N,\underline{m}}(0, a^1, ..., a^N) + O(Z_n^{\frac{s+1}{s-p}}).$$

By Proposition 4.4.1, it is possible to choose $(a^1, ..., a^N)$ to minimize $\mathsf{F}_{N,\underline{m}}$ such that the best upper bound is

$$\mathsf{E}_{Z_n}(\Omega_{t_n}) \leq \sum_{i=0}^{N} e_0(m_n^i) - Z_n \mathcal{V}(F_n^0) + Z_n^{s/(s-p)} \mu_{N,\underline{m}} + o(Z_n^{s/(s-p)}). \tag{4.5.6}$$

Lastly, an expansion of the energy E_Z up to the third-order term in Z is needed. Set F_n^0 are the sets in Lemma 4.3.1 so that

$$\mathsf{E}_{Z_n}(\Omega_n) = \sum_{i=0}^{N} e_0(m^i) - Z_n \mathcal{V}(F_n^0) + Z_n^{s/(s-p)} \mathsf{F}_{N,\underline{m}}(0, y1, ..., y_N) + o(Z_n^{s/(s-p)}).$$

Then, set $\xi_n^i = x_n^i Z_n^{1/(s-p)}$ for $i = 1, 2, ..., N$. Follow the lower bound (4.5.5), by using the upper bound (4.5.6), there exists

$$\begin{aligned} Z_n^{s/(s-p)} \mu_{N,\underline{m}} + o(Z_n^{s/(s-p)}) &\geq \mathsf{E}_{Z_n}(\Omega_n) - \sum_{i=0}^{N} e_0(m_n^i) + Z_n \mathcal{V}(F_n^0) \\ &\geq Z_n^{s/(s-p)} \mathsf{F}_{N,\underline{m}}(0, \xi_n^1, ..., \xi_n^N)(1 + o(1)). \end{aligned}$$

In conclusion, $\{\xi_n^i\}_{i=0,...,N}$ is a minimizing sequence for $\mathsf{F}_{N,\underline{m}}$. By the Proposition 4.4.1, the ξ_n^i is bounded and up to the extraction of a subsequence for each $i = 1, 2, ..., N$, $\xi_n^i \to y^i$, minimizers of $\mathsf{F}_{N,\underline{m}}$ as $n \to \infty$. Therefore, (4.2.7) is proved. The proof of Theorem 4.2.2 is complete.

□

Proof of Theorem 4.2.3

The idea to proof Theorem 4.2.3 is to make the divergent components of a minimizer of E_Z inherit the same Lagrange multiplier. Therefore, E^i satisfies the same Euler-Lagrange equation. Besides, the radius of the minimizers (when they are balls) is uniquely determined by the Lagrange multiplier. It is predictable that the equipartition of mass between the components of the generalized minimizers is true no matter the minimizers are balls or not.

By the previous proofs and lemmas, the reduced boundary $\partial^*\Omega_n$ is a disjoint union of smooth hypersurfaces. Besides, by [[5], Theorem 2.7], $\partial^*\Omega_n$ is of class $C^{3,\beta}$ for $\beta < d-1-s$. Define $v\Omega_n(x)$ as the Riesz potential such that

$$v\Omega_n(x) := \int_\Omega \frac{1}{|x-y|^s} dy.$$

Thus, the Euler-Lagrange equation

$$(d-1)\kappa(x) + 2v\Omega_n(x) - Z_n|x|^{-p} = \lambda_n \tag{4.5.7}$$

is satisfied pointwise on $\partial^*\Omega_n$, where κ is the mean curvature in $\mathbb{R}^d$ and λ_n is the Lagrange multiplier. Furthermore. from the previous proof, Ω_n is $C^{1,\alpha}$ close to the sets

$$S_n := \big[E^0 + \bigcup_{i=1}^{N}(E^i + x_n^i)\big],$$

for all fixed $R > 0$ with $E^i \Subset B_R(0)$,

$$\partial^*\tilde{\Omega}_n^i := (\partial^*\Omega_n - x_n^i) \cap B_R(0) \to \partial^* E^i \text{ in } C^{1,\alpha} \text{ for all } \alpha \in (0, \frac{1}{2}).$$

In addition, the former are expressed as graphs over the limiting sets E^i,

$$\partial^*\tilde{\Omega}_n^i := \{y = \Psi_n(x) := x + \psi_n(x)\nu_i(x) : x \in \partial^* E^i\},$$

with $\psi_n(x) \to 0$ in $C^{1,\alpha}$[[29], Theorem 4.2]. Therefore, $\partial^* E^i$ is of class $C^{3,\alpha}$ and its normal vector $\nu_{E^i} \in C^2$ by the above regularity theorem, as each E^i is itself a minimizer of E_0. Moreover, the Riesz potentials $v_{\tilde{\Omega}_n^i}$ are bounded in $C^{1,\beta}(B_R(0))$ by [[5], Proposition 2.1]. Thus, they converge uniformly to ν_{E^i} in $B_R(0)$.

As the following step, it is possible to integrate the Euler-Lagrange equation (4.5.7) by parts over $\partial^* \tilde{\Omega}_n^i$ as

$$\int_{\partial^* \tilde{\Omega}_n^i} (\mathrm{div}_{\tau_n} \zeta - (2v_{\tilde{\Omega}_n^i} - Z_n |x|^{-p})(\zeta \cdot \nu_n)) d\mathcal{H}^{d-1} = \lambda_n \int_{\partial^* \tilde{\Omega}_n^i} \zeta \cdot \nu_n d\mathcal{H}^{d-1}, \tag{4.5.8}$$

for any $\zeta \in C_0^\infty(B_R(0); \mathbb{R}^d)$, $\nu_n := \nu_{\tilde{\Omega}_n^i}$, and the tangential divergence on $\partial^* \tilde{\Omega}_n^i$ is defined as

$$\mathrm{div}_{\tau_n} \zeta = \mathrm{div} \zeta - \nu_n \cdot D\zeta \nu_n.$$

Then, integrals over $\partial^* E^i$ with Jacobian $J_n = |\det D\Psi_n|$ is obtained by using the parametrization Ψ_n. There exist $\mathrm{div}_{\tau_n} \zeta \to \mathrm{div}_{\tau_{E^i}} \zeta$ and $J_n \to 1$ as $\nu_n \to \nu_{E^i}$ by the $C^{1,\alpha}$ convergence and $V_{E^i} \in C^2$. Therefore, by passing to the limit $n \to \infty$ in both integrals in (4.5.8), there exist

$$\int_{\partial^* \tilde{\Omega}_n^i} (\mathrm{div}_{\tau_n} \zeta - (2v_{\tilde{\Omega}_n^i} - Z_n |x|^{-p})(\zeta \cdot \nu_n)) d\mathcal{H}^{d-1} \longrightarrow \int_{\partial^* E^i} (\mathrm{div}_{\tau_{E^i}} \zeta - 2v_{E^i}(\zeta \cdot \nu_{E^i})) d\mathcal{H}^{d-1},$$

and

$$\int_{\partial^* \tilde{\Omega}_n^i} \zeta \cdot \nu_n d\mathcal{H}^{d-1} \longrightarrow \int_{\partial^* E^i} \zeta \cdot \nu_{E^i} d\mathcal{H}^{d-1}.$$

As a result, for some limiting Lagrange multiplier λ_0, there exists $\lambda_n \to \lambda_0$. In addition, the values of λ_n are the same for each component of $\partial^* \Omega_n$ by (4.5.7) and the value of λ_0 is independent of $i = 0, 1, ..., N$. Then, with the same Lagrange multiplier λ_0, the limiting curvature equation is the same for each limiting set E^i. The limiting sets E^i are all balls since $s < \overline{s}(d)$ and they are all have same radius as the Lagrange multiplier is uniquely determined by the mass m^i for balls.

□

5 The Triblock Copolymers

5.1 Introduction

Nakazawa and Ohta address the theory of triblock copolymers in two dimensions. The triblock copolymer has been studied in [14] and [15]. As the following, the asymptotic behaviour of the energy functional from their theory is addressed.

$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = [-\frac{1}{2}, \frac{1}{2}]^2$ denotes the n-dimensional flat torus of unit volume. Define $u = (u_1, u_2)$ and $u_0 = 1 - u_1 - u_2$. Define the order parameters $u_i, i = 0, 1, 2$ on $\mathbb{T}^2$. Therefore, on $BV(\mathbb{T}^2; \{0,1\})$, the triblock energy is defined as

$$\mathcal{E}(u) := \frac{1}{2}\sum_{i=0}^{2}\int_{\mathbb{T}^2}|\nabla u_i| + \sum_{i,j=1}^{2}\frac{\gamma_{ij}}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} u_i(x)u_j(y)G_{\mathbb{T}^2}(x-y)dxdy \tag{5.1.1}$$

This energy can be minimized by two mass or area constraints

$$\frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} u_i = M_i, i = 1, 2. \tag{5.1.2}$$

In (5.1.1), u_i represents the relative monomer density. When $u_1 = 1$, it represents the pure-A region; when $u_2 = 1$, it represents the pure-B region; when $u_0 = 1$, it represents the pure-C region. The first term of (5.1.1) is the perimeter of the interface and second term is the long range interaction energy. γ_{ij} represents a symmetric matrix such that $\gamma = [\gamma_{ij}] \in \mathbb{R}^{2\times 2}$. In addition, in (5.1.2), M_1 represents the area fraction of type-A and M_2 represents the area fraction of type-B.

5.2 Definitions

In addition, Green's function is necessary to be introduced. $G_{\mathbb{T}^2}$, the zero-mean Green's function for $-\Delta$ on $\mathbb{T}^2$ is given by

$$-\Delta G_{\mathbb{T}^2}(\cdot - y) = \delta(\cdot - y) - 1, \text{with} \int_{\mathbb{T}^2} G_{\mathbb{T}^2}(x-y)dx = 0,$$

for each $y \in \mathbb{T}^2$ and the δ is the Dirac delta function at the origin . In two dimensions, for $\max|x-y| < 1/2$, the Green's function is given by

$$G_{\mathbb{T}^2}(x-y) = -\frac{1}{2\pi}\log|x-y| + R_{\mathbb{T}^2}(x-y), \tag{5.2.1}$$

where $R_{\mathbb{T}^2} \in C^\infty(\mathbb{T}^2)$. $R_{\mathbb{T}^2}$ is the regular part of the Green's function.

5.3 The Appropriate Droplet Scaling

For the scaling, it is very similar to the diblock case. First, the new parameter η represents the characteristic length scale of the droplet components. Then η^2 represents the areas scale. Therefore, define the mass constraints as

$$\int_{\mathbb{T}^2} u_i = \eta^2 M_i \text{ for some fixed } M_i, i = 1, 2.$$

The rescaled u_i is

$$v_{i,\eta} = \frac{u_i}{\eta^2}, i = 0, 1, 2 \text{ with } \int_{\mathbb{T}^2} u_{i,\eta} = M_i, i = 1, 2. \tag{5.3.1}$$

Besdies, with some fixed constants $\Gamma_{ij} \geq 0$, the rescaled matrix γ is

$$\gamma_{ij} = \frac{1}{|\log\eta|\eta^3}\Gamma_{ij}$$

Define

$$\Gamma_{ii} > 0, \text{with } i = 1, 2, \quad \Gamma_{12} > 0, \quad \text{and} \quad \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2 > 0$$

in this section. Define $v_\eta = (v_{1,\eta}, v_{2,\eta})$, then assume v_η is in the space

$$X_\eta := \{(v_{1,\eta}, v_{2,\eta}) | \eta^2 v_{i,\eta} \in BV(\mathbb{T}^2; \{0,1\}), v_{1,\eta}, v_{2,\eta} = 0 \ a.e.\} \tag{5.3.2}$$

since the $v_{i,\eta}$ is finite perimeter and disjoint. Therefore, define the functional

$$\mathsf{E}_\eta(v_\eta) := \frac{1}{\eta}\mathcal{E}(u) = \begin{cases} \frac{\eta}{2}\sum_{i=0}^{2}\int_{\mathbb{T}^{\neq}} |\nabla v_{i,\eta}| & \\ +\sum_{i,j=1}^{2}\frac{\Gamma_{ij}}{2|\log\eta|}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v_{i,\eta}(x)v_{j,\eta}(y)G_{\mathbb{T}^2}(x-y)dxdy & v_\eta \in X_\eta \\ \infty & \text{otherwise.} \end{cases} \tag{5.3.3}$$

Note that for large enough M_i, the above choice of parameters will cause fragmentation of a minimizing sequence $v_\eta = \sum_{k=1}^{K} v_\eta^k$ into K isolated components at a distinct point $\xi^k \in \mathbb{T}^2$. Besides, it is supported on a pair of sets $(\omega_{1,\eta}^k, \omega_{2,\eta}^k)$ with characteristic length scale $O(\eta)$, which is the result showed in the binary case. Therefore, at η-scale, define limiting profile $z_i^k := \chi_{A_i^K}$ for pairs of sets $A^k = (A_1^k, A_2^k) \in \mathbb{R}^2$, then the minimizing components is

$$v_{i,\eta}^K(\eta x + \xi^k) = \eta^{-2} z_i^k(x).$$

In addition, for $m_i^k = |A_i^k|$, the minimizer v_η may be defined as a superposition of point particles such that

$$v_\eta \rightharpoonup \sum_{k=1}^{K}(m_1^k, m_2^k)\delta_{\chi^k}$$

Moreover, for sets $A_1, A_2, \subset \mathbb{R}^2$ with $|A_1 \cap A_2| = 0$, define the perimeter of the 2-cluster $A = (A_1, A_2)$ as

$$\mathrm{Per}_F(A) = \frac{1}{2}\sum_{i=0}^{2}\mathcal{H}^1(A_i \cap F), \tag{5.3.4}$$

where $A_0 = (A_1 \cup A_2)^C$.

Therefore, the formal expansion of the energy form is

$$
\begin{aligned}
E_\eta(v_\eta) &= \sum_{k=1}^{K}\sum_{i=0}^{2}\frac{\eta}{2}\int_{\mathbb{T}^2}|\nabla v_{i,\eta}^k| + \frac{\Gamma_{ij}}{2|\log\eta|}\sum_{k,\ell=1}^{K}\sum_{i,j=1}^{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} v_{i,\eta}^k(x)v_{j,\eta}^\ell(y)G_{\mathbb{T}^2}(x-y)dxdy \\
&= \sum_{k=1}^{K}\sum_{i=0}^{2}\frac{1}{2}\int_{A_i^k}|\nabla z_i^k| + \frac{\Gamma_{ij}}{2|\log\eta|}\sum_{k,\ell=1}^{K}\sum_{i,j=1}^{2}\int_{A_i^k}\int_{A_j^\ell} G_{\mathbb{T}^2}(\xi^k+\eta\tilde{x}-\xi_\eta^\ell\tilde{y})d\tilde{x}dy \\
&= \sum_{k=1}^{K}\left(\mathrm{Per}_{\mathbb{R}^2}(A^k) + \sum_{i,j=1}^{2}\frac{\Gamma_{ij}}{4\pi}|A_i^k||A_j^k|\right) + O(|\log\eta|^{-1}).
\end{aligned}
$$

The expression $p(m_1, m_2) = \mathrm{Per}_{\mathbb{R}^2}(A)$ represents the perimeter of the minimizing cluster $A = (A_1, A_2)$ with $m_i = |A_i|$. Therefore, when both $m_i > 0, i = 1, 2$, the minimizer [31] is at a double bubble such that

$$
e_0(m) = p(m_1, m_2) + \sum_{i,j=1}^{2}\frac{\Gamma_{ij}m_im_j}{4\pi}. \tag{5.3.5}
$$

Moreover, the minimizer is single bubble when $m_1 = 0$ or $m_2 = 0$, then $p(m_1, 0) = 2\sqrt{\pi m_1}$ or $p(0, m_2) = 2\sqrt{\pi m_2}$, such that

$$
\begin{gathered}
e_0(m) = e_0(m_1, 0) = 2\sqrt{\pi m_1} + \tfrac{\Gamma_{11}(m_1)^2}{4\pi} \\
or \\
e_0(m) = e_0(0, m_2) = 2\sqrt{\pi m_2} + \tfrac{\Gamma_{22}(m_2)^2}{4\pi}.
\end{gathered} \tag{5.3.6}
$$

The minimizer of E_η will form an array of either single or double bubbles or both since the components of Ω_η has no other shape, and can be determined by the higher order terms in a detailed energy expansion. In fact, the quadratic term in e_0 may decrease when either M_i is large that total energy is reduced by splitting into smaller components. Similarly what have defined in the case of sharp interface of diblock copolymer, to splitting of sets with the large area effectively, define

$$
\overline{e_0}(M) := \inf\left\{\sum_{k=1}^{\infty} e_0(m^k) : m^k = (m_1^k, m_2^k), m_i^k \geq 0, \sum_{k=1}^{\infty} m_i^k = M_i, i = 1, 2\right\} \tag{5.3.7}
$$

5.4 Main Results

Here are some the most important results of the minimizers of $\overline{e_0}(M)$.

Theorem 5.4.1. • Finiteness Theorem: For any $M = (M_1, M_2)$ with $M_1, M_2 > 0$, a minimizing configuration for $\overline{e_0}(M)$ has finitely many nontrivial components. There exist $K < \infty$ and pairs $m^1, \ldots, m^K$, with $m^k = (m_1^k, m_2^k) \neq (0,0)$, for $\overline{e_0}(M) = \sum_{k=1}^{K} e_0(m^k)$.

Theorem 5.4.2. • Coexistence Theorem: Given K_1 and $K_2 > 0$, and $\Gamma_{12} = 0$, there exist $\overline{M_1}$ and $\overline{M_2}$ such that for all $M_1 > \overline{M_1}$ and $M_2 > \overline{M_2}$ minimizing configurations of (5.3.7) have at least K_1 double bubbles and K_2 single bubbles.

- All Single Bubbles Theorem: There exist constants M_i^* depending only on $\Gamma_{ii}, i = 1, 2$, such that for any given $M_1 > 4M_1^*, M_2 > 4M_2^*$. There exists a threshold Γ_{12}^* such that for all $\Gamma_{12} > \Gamma_{12}^*$, any minimizing configuration of (5.3.7) has no double bubbles. Moreover, all single bubbles have the same size.

- One Double Bubble Theorem: There exist constants m_i^* depending only on $\Gamma_{ii}, i = 1, 2$, such that for any given $M_i < \min\{m_i^*, \pi\Gamma_{ii}^{-2/3}\}, i = 1, 2$, and sufficiently small $\Gamma_{12} > 0$ such that
$$\frac{\Gamma_{12}}{2\pi} M_1 M_2 + p(M_1, M_2) < 2\sqrt{\pi}(\sqrt{M_1} + \sqrt{M_2}),$$
here p represents the perimeter. Therefore, there is a unique minimizer of (5.3.7) made of one double bubble.

5.5 Convergence Theorems

As a following, the first-order convergence of E_η is defined below. First, let v_η^* be the minimizers of E_η, then the global minimizers of E_η is

$$E_\eta(v_\eta^*) = \min\{E_\eta(v_\eta) | v_\eta = (v_{1,\eta}, v_{2,\eta}) \in X_\eta, \int_{\mathbb{T}^2} v_\eta = M\}, \tag{5.5.1}$$

where $\int_{\mathbb{T}^2} v_\eta = M$ is the given mass condition and X_η is defined in (5.3.2).

Theorem 5.5.1. As a result, let $v_\eta^* = \eta^{-2}\chi_{\Omega_\eta}$ be the minimizers of (5.5.1) for all $\eta > 0$, then there exists a sequence $\eta \to 0$ and $K \in \mathbb{N}$ such that:

1. there exist connected clusters $A^1, A^2, \dots, A^K$ in $\mathbb{R}^2$ and points $x_\eta^k \in \mathbb{T}^2, k = 1, 2, \dots, K$, for which

$$\eta^{-2}\Big|\Omega_\eta \triangle \bigcup_{k=1}^{K}(\eta A^k + x_\eta^k)\Big| \overset{\eta\to 0}{\longrightarrow} 0; \tag{5.5.2}$$

2. each $A^k, k = 1, 2, \dots, K$ is a minimizer of $\mathcal{G}$ such that

$$\mathcal{G}(A^k) = e_0(m^k), \quad m^k = (m_1^k, m_2^k) = |A^k|, \tag{5.5.3}$$

and

$$\overline{e_0}(M) = \lim_{\eta\to 0} E_\eta(v_\eta) = \sum_{k=1}^{K}\mathcal{G}(A^k) = \sum_{k=1}^{K} e_0(m^k). \tag{5.5.4}$$

3. $x_\eta^k \overset{\eta\to 0}{\longrightarrow} x^k, \forall k = 1, 2, \dots, K$, and $\{x^1, x^2, \dots, x^K\}$ attains the minimum of $\mathcal{F}_K(y^1, y^2, \dots, y^K; \{m^1, m^2, \dots, m^K\})$ over all $\{y^1, y^2, \dots, y^K\} \in \mathbb{T}^2$.

Moreoever, comparing to the diblock case, this theorem provides a better description of energy minimizers with more details.

In addition, the limit of Γ-convergence is defined below. First, define a class of measures with countable support on $\mathbb{T}^2$ such that

$$Y := \Big\{v_0 = \sum_{k=1}^{\infty}(m_1^k, m_2^k)\delta_{x^k} \mid m_i^k \geq 0, x^k \in \mathbb{T}^2 \text{ distinct points}\Big\}.$$

Therefore, the functional on Y is

$$E_0(v_0) := \begin{cases} \sum_{k=1}^{\infty} \overline{e_0}(m^k), & \text{if } v_0 \in Y \\ \infty & \text{otherwise.} \end{cases} \tag{5.5.5}$$

Theorem 5.5.2. As a result, the first Γ-convergence theorem is defined:

$$E_\eta \xrightarrow{\Gamma} E_0 \quad \text{as} \quad \eta \to 0.$$

That is,

1. Let $v_\eta \in X_\eta$ be a sequence with $\sup_{\eta>0} E_\eta(v_\eta) < \infty$. There exists a sequence $v_\eta \to v_0$ and $v_0 \in Y$ such that
$$\liminf_{\eta\to 0} E_\eta(v_\eta) \geq E_0(v_0).$$

2. Let $v_0 \in Y$ with $E_0(v_0) < \infty$. There exists a sequence $v_\eta \to v_0$ weakly as measures such that
$$\limsup_{\eta\to 0} E_\eta(v_\eta) \leq E_0(v_0).$$

Furthermore, for the second Γ-convergence theorem, it is at the level of $|\log\eta|^{-1}$ in the energy. It represents the interaction energy between components at the minimal energy $\overline{e_0}(M)$. Therefore, define

$$F_\eta(v_\eta) := |\log\eta|\Big[E_\eta(v_\eta) - \overline{e_0}(\int_{\mathbb{T}^2} v_\eta)\Big], \quad v_\eta \in X_\eta \tag{5.5.6}$$

As the similar process in the binary case, for $K \in \mathbb{N}, m_1^k \geq 0, m_2^k \geq 0$ and $(m_1^k)^2 + (m_2^k)^2 > 0$, the sequence $K \otimes (m_1^k, m_2^k)$ is

$$(K \otimes (m_1^k, m_2^k))^k := \begin{cases} (m_1^k, m_2^k), & 1 \leq k \leq K, \\ (0,0), & K+1 \leq k < \infty \end{cases}$$

Therefore, define $\mathcal{M}_M$ as the set of optimal sequences of all clusters for (5.3.7)

$$\mathcal{M}_M := \Big\{K \otimes (m_1^k, m_2^k) : K \otimes (m_1^k, m_2^k) \text{ minimizes (5.3.7) for } M_i, i = 1,2, \\ \text{and } \overline{e_0}(m^k) = e_0(m^k), m^k = (m_1^k, m_2^k)\Big\}.$$

Then, the limiting energy functional F_0 can be defined as

$$F_0(v_0) := \begin{cases} \sum_{i,j=1}^{2} \frac{\Gamma_{ij}}{2}\Big\{ \sum_{k=1}^{K}[f(m_i^k, m_j^k) + m_i^k m_j^k R_{\mathbb{T}^2}(0)]+ & \text{if } v_0 = \displaystyle\sum_{k=1}^{K} m^k \delta_{x^k}, \{x^1, \ldots, x^K\} \\ \sum_{\substack{k,\ell=1 \\ k\neq\ell}}^{K} m_i^k m_j^\ell G_{\mathbb{T}^2}(x_i^k - x_j^\ell)\Big\} & \text{with distinct points in } \mathbb{T}^2 \text{ and} \\ & K \otimes m^k \in \mathcal{M}_M \\ \infty & \text{otherwise,} \end{cases} \tag{5.5.7}$$

where

$$f(m_i^k, m_j^k) = \frac{1}{2\pi} \int_{A_i^k} \int_{A_j^k} \log \frac{1}{|x-y|} dxdy$$

and A^k are the minimizers of $e_0(m^k)$ and defined in the first Γ-limit.

Theorem 5.5.3. As a result,

$$F_\eta \xrightarrow{\Gamma} F_0 \quad \text{as} \quad \eta \to 0.$$

Condition 1 and Condition 2 of Theorem 5.5.2 are still hold with the replacing of E_η and E_0 with F_η and F_0.

5.6 Geometric Properties of Global Minimizers

In this section, the geometric properties of global minimizers of $\overline{e_0}(M)$ is described [16]. The following lemmas help to overcome the difficulty which there is not such simple formula for the double bubbles. Besides, for the single bubble case, the prove proofs are analogous mostly and much easier.

Lemma 5.6.1. Since

$$\frac{\partial}{\partial m_1} p(m_1, m_2) = \lim_{\varepsilon \to 0^+} \frac{p(m_1+\varepsilon, m_2) - p(m_1, m_2)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{p(m_1, m_2) - p(m_1-\varepsilon, m_2)}{\varepsilon},$$

then $p(m_1 \pm -\varepsilon, m_2)$ is needed to bound.

Here are some definitions that need to be introduced [16]. Let B represents the double bubbles with masses (m_1, m_2). Let C_i represents the circular arc of the boundary of the lobe with the mass m_i, radius r_i, and center O_i, which $i = 1, 2$. Let C_0 represents the central arc by P, which is one of the triple junction points by the tangent lines τ_i to C_i at P and $i = 0, 1, 2$. In addition, the angle between two tangent lines τ_i and τ_j with $i \neq j$ is $2\pi/3$. Besides, let $T_t(C_1)$ represents the scaling of C_1 and center at O_1, and $t > 0$ is the ratio.

Upper Bound

Starting with the upper bound of the double bubble, which is $p(m_1+\varepsilon, m_2)$, and only describe near P since it is the same construction near the other triple $\widetilde{P}$. Please see Figure 2 [16].

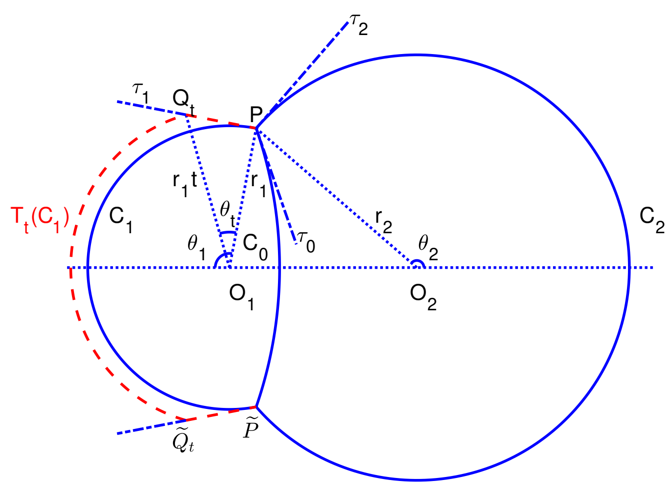

Figure 2: Construction for the upper bound of $p(m_1 + \varepsilon, m_2)$.

First, set $T_t(C_1)$ with $t = 1+\delta$ to enlarge C_1 for some $\delta = \delta(\varepsilon)$. Then, connect the triple junction point $P \in C_0 \cup C_2$ to $T_t(C_1)$ with the segment $S_t := \overline{PQ_t}$, where $Q_t := T_t(C_1) \cap \tau_1$. For the other triple junction point $\widetilde{P}$, the process is very similar that connect to $T_t(C_1)$ with the segment $\widetilde{S}_t := \overline{\widetilde{P}\widetilde{Q}_t}$, where $\widetilde{Q}_t$ is the reflection of Q_t with respect to $\overline{O_1O_2}$.

Moreover, set an admissible competitor B_t with mass $x + \varepsilon$ of type I constituent and mass m_2 of type II constituent, and does not need to be a double bubble. Therefore, B_t is the region inside

$$B_t := C_0 \cup C_2 \cup \widehat{QQ_t} \cup S_t \cup \widetilde{S}_t.$$

Set $\theta_t := \angle PO_1Q_t$ and the triangle $\triangle PO_1Q_t$ satisfies

$$|O_1 - Q_t| = r_1 t, \quad |O_1 - P| = r_1, \quad \cos\theta_t = \frac{|O_1 - P|}{|O_1 - Q_t|} = \frac{1}{t}, \quad \mathcal{H}^1(S_t) = r_1 \tan\theta_t.$$

Therefore, $\theta_t = \sqrt{2\delta} + o(\sqrt{\delta})$ when $t = 1 + \delta$ with $0 < \delta \ll 1$ since

$$\cos\theta_t = 1 - \frac{(\theta_t)^2}{2} + O((\theta_t)^4) = \frac{1}{1+\theta} = 1 - \theta + o(\delta).$$

Since the arc length of C_1 in $\triangle PO_1Q_t$ is $r_1\theta_t$, thus

$$|\mathcal{H}^1(S_t) - \mathcal{H}^1(C_1 \cap \triangle PO_1Q_t)| = r_1(\tan\theta_t - \theta_t) = r_1\Big(\frac{(\theta_t)^3}{3} + O((\theta_t)^5)\Big) = O(\delta\sqrt{\delta}),$$

and

$$|\mathcal{H}^1(\widetilde{S}_t) - \mathcal{H}^1(C_1 \cap \triangle \widetilde{P}O_1\widetilde{Q}_t)| = O(\delta\sqrt{\delta}).$$

Therefore, the difference in perimeter is

$$\begin{aligned}
&\mathcal{H}^1(\partial B_t) - \mathcal{H}^1(\partial B) \\
&= \Big[\mathcal{H}^1(\widehat{QQ_t}) + \mathcal{H}^1(S_t) + \mathcal{H}^1(\widetilde{S}_t) + \mathcal{H}^1(C_0) + \mathcal{H}^1(C_2)\Big] - \Big[\mathcal{H}^1(C_1) + \mathcal{H}^1(C_0) + \mathcal{H}^1(C_2)\Big] \\
&= 2r_1(1+\delta)(\theta_1 - \theta_t) - 2r_1(\theta_1 - \theta_t) + O(\delta\sqrt{\delta}) \\
&= 2\theta_1 r_1 \delta + O(\delta\sqrt{\delta}).
\end{aligned}$$

Therefore, the estimated difference in area is

$$\begin{aligned}
&\mathcal{H}^2(B_t) - \mathcal{H}^2(B) \\
&= (\theta_1 - \theta_t) r_1^2 [(1+\delta)^2 - 1] + 2\Big[\mathcal{H}^2(\triangle P O_1 Q_t) - \frac{\theta_t r_1^2}{2}\Big] \\
&= 2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta}) + r_1^2(\tan\theta_t - \theta_t) \\
&= 2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta}).
\end{aligned}$$

As a result, the difference in area between the competitor B_t and the original double bubble B is

$$2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta})$$

Since the difference above has to be equal to ε, then

$$\delta = \frac{\varepsilon}{2\theta_1 r_1^2} + o(\varepsilon),$$

and

$$\lim_{\varepsilon\to 0^+} \frac{p(m_1+\varepsilon, m_2) - p(m_1, m_2)}{\varepsilon} \leq \lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^1(\partial B_t) - p(m_1, m_2)}{\varepsilon} = \lim_{\varepsilon\to 0^+} \frac{2\theta_1 r_1 \delta + O(\delta\sqrt{\delta})}{\varepsilon} = \frac{1}{r_1}.$$

Lower Bound

Furthermore, the set up for the lower bound is very similar as the process of upper bound. Starting with the lower bound of the double bubble, which is $p(m_1-\varepsilon, m_2)$, and only describe near P since it is the same construction near the other triple $\widetilde{P}$. Please see Figure 3 [16].

First, set $T_t(C_1)$ with $t = 1-\delta$ to shrink C_1 for some $\delta = \delta(\varepsilon)$. Then, connect the triple junction point $P \in C_0 \cup C_2$ to $T_t(C_1)$ with the segment $S_t := \overline{PQ_t}$ that is tangent to $T_1(C_1)$ at Q_t. For the other triple junction point $\widetilde{P}$, the process is very similar that connect to $T_t(C_1)$ with the segment $\widetilde{S}_t := \overline{\widetilde{P}\widetilde{Q}_t}$, where $\widetilde{Q}_t$ is the reflection of Q_t with respect to $\overline{O_1 O_2}$.

Moreover, set an admissible competitor B_t with mass $x+\varepsilon$ of type I constituent and

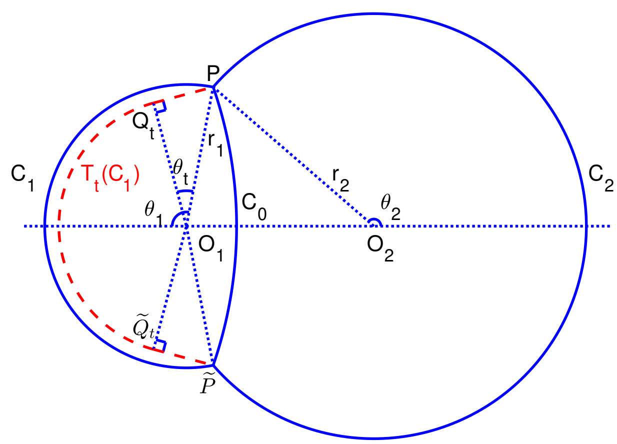

Figure 3: Construction for the lower bound of $p(m_1 - \varepsilon, m_2)$.

mass m_2 of type II constituent, and does not need to be a double bubble. Therefore, B_t is the region inside

$$B_t := C_0 \cup C_2 \cup \widehat{QQ_t} \cup S_t \cup \widetilde{S}_t.$$

Set $\theta_t := \angle PO_1Q_t$ and the triangle $\triangle PO_1Q_t$ satisfies

$$|O_1 - Q_t| = r_1 t, \quad \theta_t = \arccos \frac{|O_1 - Q_t|}{|O_1 - P|} = t, \quad \mathcal{H}^1(S_t) = r_1 \sin \theta_t.$$

Therefore, $\theta_t = \sqrt{2\delta} + o(\sqrt{\delta})$ when $t = 1 - \delta$. Therefore, the difference in perimeter is

$$\begin{aligned}
&\mathcal{H}^1(\partial B) - \mathcal{H}^1(\partial B_t)\\
=&\Big[\mathcal{H}^1(C_1) + \mathcal{H}^1(C_0) + \mathcal{H}^1(C_2)\Big] - \Big[\mathcal{H}^1(\widehat{QQ_t}) + \mathcal{H}^1(S_t) + \mathcal{H}^1(\widetilde{S}_t) + \mathcal{H}^1(C_0) + \mathcal{H}^1(C_2)\Big]\\
=&\, 2\theta_1 r_1 - 2(\theta_1 - \theta_t) r_1 (1 - \delta) - 2 r_1 \sin \theta_t + O(\delta\sqrt{\delta})\\
=&\, 2\theta_1 r_1 \delta + O(\delta\sqrt{\delta}).
\end{aligned}$$

Therefore, the estimated difference in area is

$$\begin{aligned}
&\mathcal{H}^2(B) - \mathcal{H}^2(B_t) \\
&= (\theta_1 - \theta_t) r_1^2 [(1 - 1 - \delta)^2] + 2\Big[\frac{\theta_t r_1^2}{2} - \mathcal{H}^2(\triangle P O_1 Q_t)\Big] \\
&= 2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta}) + r_1^2(\theta_t - sin\theta_t \cos\theta_t) \\
&= 2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta}).
\end{aligned}$$

As a result, the difference in area between the competitor B_t and the original double bubble B is

$$2\theta_1 r_1^2 \delta + O(\delta\sqrt{\delta})$$

Since the difference above has to be equal to ε, then

$$\delta = \frac{\varepsilon}{2\theta_1 r_1^2} + o(\varepsilon),$$

and

$$\lim_{\varepsilon \to 0^+} \frac{p(m_1, m_2) - p(m_1 - \varepsilon, m_2)}{\varepsilon} \geq \lim_{\varepsilon \to 0^+} \frac{p(m_1, m_2) - \mathcal{H}^1(\partial B_t)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{2\theta_1 r_1 \delta + O(\delta\sqrt{\delta})}{\varepsilon} = \frac{1}{r_1}.$$

As a result, it holds

$$\frac{\partial}{\partial m_i} p(m_1, m_2) = \frac{1}{r_i}, \quad i = 1, 2,$$

where $r_i = r_i(m_1, m_2)$.

Consider two arbitrary different double bubbles B_k, B_h and set $x_k := m_1^k, y_k := m_2^k$ represents the masses of the two lobes of B_k. Then there exists

$$\begin{aligned}
e_0(x_k + \varepsilon, y_k) - e_0(x_k, y_k) &= \varepsilon \frac{\partial e_0(x_k, y_k)}{\partial x_k} + \frac{\varepsilon^2}{2} \frac{\partial^2 e_0(x_k, y_k)}{\partial x_k^2} + O(\varepsilon^3), \\
e_0(x_h - \varepsilon, y_h) - e_0(x_h, y_h) &= -\varepsilon \frac{\partial e_0(x_h, y_h)}{\partial x_h} + \frac{\varepsilon^2}{2} \frac{\partial^2 e_0(x_h, y_h)}{\partial x_h^2} + O(\varepsilon^3),
\end{aligned}$$

since the minimality of $\mathcal{B}$ gives the necessary condition as

$$0 \leq e_0(x_k+\varepsilon, y_k) + e_0(x_h-\varepsilon, y_h) + \sum_{j\geq 1, j\neq k,h} e_0(x_j,y_j) - \sum_{j\geq 1} e_0(x_j,y_j)$$
$$= \varepsilon\Big(\frac{\partial e_0(x_k,y_k)}{\partial x_k} - \frac{\partial e_0(x_h,y_h)}{\partial x_h}\Big) + \frac{\varepsilon^2}{2}\Big(\frac{\partial^2 e_0(x_k,y_k)}{\partial x_k^2} + \frac{\partial^2 e_0(x_h,y_h)}{\partial x_h^2}\Big) + O(\varepsilon^3).$$

Therefore,

$$\frac{\partial e_0(x_k,y_k)}{\partial x_k} = \frac{\partial e_0(x_h,y_h)}{\partial x_h}, \quad \frac{\partial^2 e_0(x_k,y_k)}{\partial x_k^2} + \frac{\partial^2 e_0(x_h,y_h)}{\partial x_h^2} \geq 0, \forall k \neq h$$

since the arbitrariness of ε. In addition, the pure second derivative in y_k is analogous.

Lemma 5.6.2. As a result, there are at least two double bubbles in an arbitrary minimizing configuration $\mathcal{B}$ of (5.3.7) represented by $B_k, k = 1, 2, \cdots$. Set m_1^k and m_2^k represent the masses of the two lobes of B_k. Then the pure second derivatives satisfy

$$\frac{\partial^2 e_0(m_1^k, m_2^k)}{\partial (m_1^k)^2}, \frac{\partial^2 e_0(m_1^h, m_2^h)}{\partial (m_2^h)^2} \geq 0$$

for all except at most one such index k respect to h.

Moreover, by Lemma 5.6.1, set $r_1 = r_1(m_1, m_2)$, then there exist

$$\frac{\partial e_0(m_1,m_2)}{\partial m_1} = \frac{\Gamma_{11}m_1 + \Gamma_{12}m_2}{2\pi} + \frac{1}{r_1}, \quad \frac{\partial^2 e_0(m_1,m_2)}{\partial m_1^2} = \frac{\Gamma_{11}}{2\pi} + \frac{\partial}{\partial m_1}\frac{1}{r_1}.$$

Therefore, it is necessary to show that

$$\lim_{m_1\to 0} \frac{\partial}{\partial m_1}\frac{1}{r_1} = -\infty, \tag{5.6.1}$$

since there exists a threshold m_1^* such that

$$\frac{\partial}{\partial m_1}\frac{1}{r_1} < -\frac{\Gamma_{11}}{2\pi},$$

for any $m_1 < m_1^*$. Thus, consider an asymmetric double bubble bounded by three circular arcs of radii r_0, r_1, r_2 with $m_1 < m_2$ as shown in Figure 4 [16].

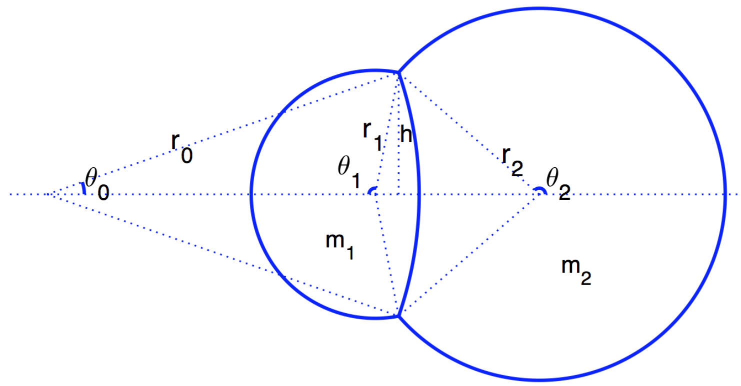

Figure 4: An asymmetric double bubble with radii r_i and half-angles θ_i, $i = 0, 1, 2$.

Note that r_0 is the radius of the common boundary of the two lobes of the double bubbles. θ_0 is half of the angle associated with the middle arc. h is the half of the distance between two triple junction points. The following equations hold [32]

$$m_1 = r_1^2(\theta_1 - \cos\theta_1 \sin\theta_1) + r_0^2(\theta_0 - \cos\theta_0 \sin\theta_0), \tag{5.6.2}$$

$$m_2 = r_2^2(\theta_2 - \cos\theta_2 \sin\theta_2) + r_0^2(\theta_0 - \cos\theta_0 \sin\theta_0), \tag{5.6.3}$$

$$h_0 = r_0 \sin\theta_0 = r_1 \sin\theta_1 = r_2 \sin\theta_2, \tag{5.6.4}$$

$$(r_0)^{-1} = (r_1)^{-1} - (r_2)^{-1}, \tag{5.6.5}$$

$$0 = \cos\theta_1 + \cos\theta_2 + \cos\theta_0, \tag{5.6.6}$$

where r_0, r_1, r_2 and $\theta_0, \theta_1, \theta_2$ are the half-angles associated with the three arcs that depended on m_1 and m_2 implicitly. Combine (5.6.4) and (5.6.5), then

$$\sin\theta_1 - \sin\theta_2 - \sin\theta_0 = 0. \tag{5.6.7}$$

Combine (5.6.6) and (5.6.7), then

$$\cos(\theta_1+\theta_0) = -\frac{1}{2},\ \cos(\theta_2-\theta_0) = -\frac{1}{2}. \tag{5.6.8}$$

Therefore,

$$\theta_1 = \frac{2\pi}{3} - \theta_0,\ \theta_2 = \frac{2\pi}{3} + \theta_0. \tag{5.6.9}$$

Considering the case $m_1 \to 0$, it implies $h \to 0$ and $r_2 \to \sqrt{m_2/\pi}$. Therefore, $\theta_2 \to \pi$ since $\theta_0, \theta_1 \to \pi/3$ so that $\theta_0 = \pi/3 - \varepsilon, \theta_1 = \pi/3 + \varepsilon, \theta_2 = \pi - \varepsilon$. Therefore, from (5.6.4), there exists

$$h = r_2 \sin(\pi-\varepsilon) = r_1 \sin(\pi/3+\varepsilon) = r_0 \sin(\pi/3-\varepsilon). \tag{5.6.10}$$

Then,

$$r_1 = r_2\frac{\sin\varepsilon}{\sin(\pi/3+\varepsilon)}, r_0 = r_2\frac{\sin\varepsilon}{\sin(\pi/3-\varepsilon)}. \tag{5.6.11}$$

As a result, (5.6.2) and (5.6.3) can be written as

$$\begin{aligned}
m_2 &= r_2^2\Big[\pi - \varepsilon + \frac{1}{2}\sin(2\varepsilon) - \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big)\Big]\\
m_1 &= r_2^2\Big[\frac{\sin^2\varepsilon}{\sin^2(\pi/3+\varepsilon)}\Big(\frac{\pi}{3} + \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} + 2\varepsilon)\Big) + \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big)\Big]\\
&= m_2\frac{\frac{\sin^2\varepsilon}{\sin^2(\pi/3+\varepsilon)}\Big(\frac{\pi}{3} + \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} + 2\varepsilon)\Big) + \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big)}{\pi - \varepsilon + \frac{1}{2}\sin(2\varepsilon) - \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big)}
\end{aligned} \tag{5.6.12}$$

Therefore, set

$$\begin{aligned}
N(\varepsilon) :=& \frac{\sin^2\varepsilon}{\sin^2(\pi/3+\varepsilon)}\Big(\frac{\pi}{3} + \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} + 2\varepsilon)\Big) + \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big)\\
=& O(\varepsilon^2),\\
D(\varepsilon) :=& \pi - \varepsilon + \frac{1}{2}\sin(2\varepsilon) - \frac{\sin^2\varepsilon}{\sin^2(\pi/3-\varepsilon)}\Big(\frac{\pi}{3} - \varepsilon - \frac{1}{2}\sin(\frac{2\pi}{3} - 2\varepsilon)\Big) = \pi + O(\varepsilon^2).
\end{aligned}$$

Then, $m_1 = \frac{N(\varepsilon)}{D(\varepsilon)} m_2$. Thus,

$$\frac{1}{m_2}\frac{dm_1}{d\varepsilon} = \frac{N'(\varepsilon)D(\varepsilon) - D'(\varepsilon)N(\varepsilon)}{D(\varepsilon)^2} = \frac{16}{3\pi}\Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big)\varepsilon + O(\varepsilon^2),$$

where

$$N'(\varepsilon) = \frac{16}{3}\Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4})\varepsilon + O(\varepsilon^2),\ D'(\varepsilon) = \frac{8}{3}\Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big)\varepsilon + O(\varepsilon^2).$$

Furthermore, it is possible to compute the derivative $\frac{\partial r_1}{\partial \varepsilon}$ from (5.6.11) and (5.6.12) that

$$r_2^1 = r_2^2 \frac{\sin^2\varepsilon}{\sin^2(\pi/3+\varepsilon)} = \frac{\sin^2\varepsilon}{\sin^2(\pi/3+\varepsilon)D(\varepsilon)} m_2,$$

then

$$\frac{1}{m_2}\frac{\partial r_1^2}{\partial \varepsilon} = \frac{\sin 2\varepsilon}{\sin^2(\pi/3+\varepsilon)D(\varepsilon)} - \frac{2\sin^2\varepsilon\cos(2\pi/3+\varepsilon)}{\sin^3(\pi/3+\varepsilon)D(\varepsilon)} - \frac{\sin^2\varepsilon D'(\varepsilon)}{\sin^2(\pi/3+\varepsilon)D^2(\varepsilon)} = \frac{8}{3\pi}\varepsilon + O(\varepsilon^2).$$

Therefore,

$$\frac{\partial r_1}{\partial \varepsilon} = \frac{1}{2r_1}\frac{\frac{\partial r_1^2}{\partial \varepsilon}}{\frac{dm_1}{d\varepsilon}} = \frac{1}{2r_1}\frac{\frac{8}{3\pi}\varepsilon + O(\varepsilon^2)}{\frac{16}{3\pi}\Big(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big)\varepsilon + O(\varepsilon^2)} \geq \frac{C}{r_1} > 0,$$

for all sufficiently small $\varepsilon < \varepsilon_0$ with C where ε_0 is a universal constants independent of Γ_{ij} and $M_i, i, j = 1, 2$. Since $\theta_1 \in (\pi/3, 2\pi/3)$ in a double bubble, then,

$$\frac{\pi}{3} r_1^2 \leq m_1 \leq \pi r_1^2.$$

Therefore, there exists another constant $C' > 0$ such that

$$\frac{\partial}{\partial m_1}\frac{1}{r_1} = -\frac{1}{r_1^2}\frac{\partial r_1}{\partial m_1} \leq -\frac{C'}{m_1^{3/2}}$$

as $m_1 \to 0$. Besides, note that

$$\frac{\partial^2 e_0(m_1, m_2)}{\partial m_1^2} = \frac{\Gamma_{11}}{2\pi} + \frac{\partial}{\partial m_1}\frac{1}{r_1} \leq \frac{\Gamma_{11}}{2\pi} - \frac{C'}{m_1^{3/2}}.$$

Lemma 5.6.3. As a result, there exist constant m_i^* such that

$$\frac{\partial^2 e_0(m_1, m_2)}{\partial m_i^2} < 0, \text{ for all } m_i < m_i^*, i = 1, 2,$$

with given Γ_{ii} and m_i^* only depends on Γ_{ii}.

In addition, consider a one single bubble with mass m of type I material. Replacing it with with two single bubbles with mass $m/2$. Then, the energy will be changed by

$$\Delta = 2\Big[\frac{\Gamma_{11}m^2}{16\pi} + \sqrt{2m\pi}\Big] - \Big[\frac{\Gamma_{11}m^2}{4\pi} + 2\sqrt{m\pi}\Big] = -\frac{\Gamma_{11}m^2}{8\pi} + 2\sqrt{m\pi}(\sqrt{2} - 1).$$

Therefore, the minimality of $\mathcal{B}$ needs $\Delta \geq 0$, which means if and only if

$$m^{3/2} \leq \frac{16\pi\sqrt{\pi}(\sqrt{2} - 1)}{\Gamma_{11}},$$

where $\mathcal{B}$ is the minimizing configuration of (5.3.7). Moreover, consider a lobe of a double bubble with mass m of type I material. Replacing it with two single bubbles with mass $m/2$. Then, the energy will be changed by

$$\begin{aligned}\Delta &= 2\Big[\frac{\Gamma_{11}m^2}{16\pi} + \sqrt{2m\pi}\Big] + \frac{\Gamma_{22}m_2^2}{4\pi} + 2\sqrt{m_2\pi} - \Big[\frac{\Gamma_{11}m^2 + 2\Gamma_{12}mm_2 + \Gamma_{22}m_2^2}{4\pi} + p(m_1, m_2)\Big] \\ &\leq -\frac{\Gamma_{11}m^2}{8\pi} + 2\sqrt{2m\pi}.\end{aligned}$$

Therefore, the minimality of $\mathcal{B}$ needs $\Delta \geq 0$, which means if and only if

$$m^{3/2} \leq \frac{16\pi\sqrt{2\pi}}{\Gamma_{11}}.$$

For type II material, the progress is analogous.

Lemma 5.6.4. As a result, there is no single bubble nor lobe of double bubbles of i-th constituent in a minimizing configuration of (5.3.7), having mass greater than

$$M*_i := \frac{8\pi}{\Gamma_{ii}^{2/3}},\ i = 1, 2.$$

Afterwards, consider there is a minimizing configuration with at least two single bubbles of type I material with mass $m_1^k = x \leq y = m_1^l$. Replacing it with one single bubble with mass $x + y$. Then, the energy will be changed by

$$\begin{aligned}\Delta &= \frac{\Gamma_{11}(x+y)^2}{4\pi} + 2\sqrt{\pi(x+y)} - \Big[\frac{\Gamma_{11}(x^2+y^2)}{4\pi} + 2\sqrt{\pi}(\sqrt{x}+\sqrt{y})\Big] \\ &= -\frac{\Gamma_{11}xy}{2\pi} + 2\sqrt{\pi}(\sqrt{x+y} - \sqrt{x} - \sqrt{y}).\end{aligned}$$

Therefore, the minimality of the minimizing configuration needs $\Delta \geq 0$, which means if and only if

$$\frac{\Gamma_{11}xy}{2\pi} \geq 2\sqrt{\pi}(\sqrt{x} + \sqrt{y} - \sqrt{x+y}).$$

Then,

$$\frac{\Gamma_{11}xy}{4\pi\sqrt{\pi}} \geq \frac{2\sqrt{xy}}{\sqrt{x}+\sqrt{y}+\sqrt{x+y}} \geq \frac{\sqrt{xy}}{\sqrt{x}+\sqrt{y}} \overset{(x\leq y)}{\geq} \frac{\sqrt{x}}{2}.$$

Besides, from the above lemma, it gives $x, y \leq M_1^*$ and follows

$$\frac{\Gamma_{11}\sqrt{x}M_1^*}{2\pi\sqrt{\pi}} \geq \frac{\Gamma_{11}\sqrt{xy}}{2\pi\sqrt{\pi}} \geq 1.$$

Thus, there exists a constant

$$\overline{m_1}^s := 4\pi^3/(\Gamma_{11}M_1^*)^2,\ y \geq x \geq \overline{m_1}^s.$$

Lemma 5.6.5. As a result, at most one single bubble of i-th constituent in a minimizing configuration has mass $m_i^k < \overline{m_1}^s$, where the constant $\overline{m_1}^s > 0, i = 1, 2$ and only depends on

Γ_{ii}.

In addition, if there exists any other double bubbles when there exists at most one double bubble with lobe of i-th constituent with mass less than $m_i^*, i = 1, 2$, then any remaining double bubbles $B_k = (m_1^k, m_2^k)$ satisfying $m_1^k \geq m_1^*$ and $m_2^k \geq m_2^*$. The energy will change by replacing a double bubbles into two single bubbles as

$$\begin{aligned}\Delta &= \sum_{i=1}^{2} \frac{\Gamma_{ii}(m_i^k)^2}{4\pi} + 2\sqrt{\pi}(\sqrt{m_1^k} + \sqrt{m_2^k}) - \Big[\frac{\Gamma_{12} m_1^k m_2^k}{2\pi} + \sum_{i=1}^{2} \frac{\Gamma_{ii}(m_i^k)^2}{4\pi} + p(m_1^k, m_2^k)\Big] \\ &\leq 2\sqrt{\pi}(\sqrt{m_1^k} + \sqrt{m_2^k}) - \frac{\Gamma_{12} m_1^k m_2^k}{2\pi}.\end{aligned}$$

When M_i^* and m_i^* depend only on $\Gamma_{ii}, i = 1, 2$ as

$$\Gamma_{12} \leq \frac{4\pi\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*})}{m_1^* m_2^*},$$

then by combining the minimalaity of the minimizing configuration and Lemma 5.6.4, the following holds

$$0 \leq \Delta \leq 2\sqrt{\pi}(\sqrt{m_1^k} + \sqrt{m_2^k}) - \frac{\Gamma_{12} m_1^k m_2^k}{2\pi} \leq 2\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*}) - \frac{\Gamma_{12} m_1^* m_2^*}{2\pi}. \quad (5.6.13)$$

Therefore, there is not any double bubble with masses $m_1^k \geq m_1^*$ and $m_2^k \geq m_2^*$ with the above Γ_{12} because the splitting will decrease the energy.

Lemma 5.6.6. As a result, any minimizing configuration of (5.3.7) has at most two double bubbles, and

$$\Gamma_{12} > \frac{4\pi\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*})}{m_1^* m_2^*}, \ \Gamma_{11} > 0, \Gamma_{22} > 0, M_1 > 0, M_2 > 0.$$

Lemma 5.6.7. If any minimizing configuration of (5.3.7) has only single bubbles, then there are finite single bubbles and all the single bubbles are of the same size with the given $\Gamma_{11}, \Gamma_{12}, \Gamma_{22}, M_1$, and M_2. The proof of this lemma follows the binary case [[17], Lemma 6.2].

5.7 Proof

Proof of Theorem 5.4.1

In conclusion, by combining Lemma 5.6.2 and Lemma 5.6.3, there exists at most one double bubble whose lobe of the i-th constituent has mass less than $m_i^*, i = 1, 2$ for any minimizing configuration for $\overline{e_0}(M)$. As a result, there exist

$$2 + \min\left\{\frac{M_1}{m_1^*}, \frac{M_2}{m_2^*}\right\}$$

at most double bubbles.

$\square$

Proof of Theorem 5.4.2 (Coexistence)

Based on the research of [31], two single bubbles of different types are more costly than a double bubbles of the same masses. When $\Gamma_{12} = 0$, the (5.3.5) will be

$$e_0(m) = p(m_1, m_2) + \frac{\Gamma 11_(m_1)^2}{4\pi} + \frac{\Gamma_{22}(m_2)^2}{4\pi}. \tag{5.7.1}$$

Then, (5.3.6) would be

$$e_0(m_1, 0) + e_0(0, m_2) = 2\sqrt{\pi m_1} + 2\sqrt{\pi m_2} + \frac{\Gamma_{11}(m_1)^2}{4\pi} + \frac{\Gamma_{22}(m_2)^2}{4\pi}. \tag{5.7.2}$$

if there are two single bubbles with different constituents types. Therefore, all single bubbles must be the same type of constituent. There are two different situations needed to discuss.

- Case I: Choosing $M_1 = K_1 M_1^*$ (M_i^* is defined in Lemma 5.6.4 if there exists any single bubbles and all single bubbles are of type II constituent. Combining Lemma 5.6.2, Lemma 5.6.3 and Lemma 5.6.4, there is at most one double bubble's lobe of type I constituent has mass less than m_1^* and other double bubbles' lobes of type I constituent must have mass between m_1^* and M_1^*. Therefore, there are at least K_1 double bubbles. Set K_d be the total number of double bubbles and $K_d < 1 + M_1/m_1^*$. Combining Lemma 5.6.2, Lemma 5.6.3 and Lemma 5.6.4, there is at most one double bubble's lobe of type II constituent has mass less than m_2^* and other double bubbles' lobes of type II constituent must have mass between m_2^* and M_2^*. Thus,

$$M_2 \geq (1 + M_1/m_1^*)M_2^* + K_2M_2^* = (1 + (K_1M_1^*)/m_1^*)M_2^* + K_2M_2^*,$$

which means that $K_dM_2^*$ represents the type II constituent is used by all double bubbles. Therefore, all the remaining type II constituent have to go into single bubbles and there are at least K_2 single bubbles.

- Case II: Choosing $M_2 = K_2M_2^*$ (M_i^* is defined in Lemma 5.6.4) if there exists any single bubbles and all single bubbles are of type I constituent. Thus, by the similar progress,

$$M_2 \geq (1 + M_2/m_2^*)M_1^* + K_1M_1^* = (1 + (K_2M_2^*)/m_2^*)M_1^* + K_1M_1^*.$$

Then, choose

$$\begin{aligned}\overline{M_1} &\geq \max\left\{K_1M_1^*, \left(1 + \frac{K_2M_2^*}{m_2^*}\right)M_1^* + K_1M_1^*\right\}\\ &= \left(1 + \frac{K_2M_2^*}{m_2^*}\right)M_1^* + K_1M_1^*, \ \forall\ M_1 \geq \overline{M_1}\end{aligned}$$

$$\begin{aligned}\overline{M_2} &\geq \max\left\{K_2M_2^*, \left(1 + \frac{K_1M_1^*}{m_1^*}\right)M_2^* + K_2M_2^*\right\}\\ &= \left(1 + \frac{K_1M_1^*}{m_1^*}\right)M_2^* + K_2M_2^*, \ \forall\ M_2 \geq \overline{M_2}.\end{aligned}$$

As a result, the minimizing configuration of (5.3.7) have at least K_1 double bubbles and K_2 single bubbles.

$\square$

Proof of Theorem 5.4.2 (All Single Bubbles)

The main idea to prove this theorem is to prove that there is a constant $\overline{m_i}^d > 0$, which only depends on Γ_{ii} that any lobe of i-th constituent of a double bubbles in the minimizing configuration of (5.3.7) has mass at least $\overline{m_i}^d > 0$.

Consider a double bubble $D = (x, m_2)$ with $M_1 > 4M_1^*$. Then, combined with Lemma 5.6.6 , there exist at least two double bubbles of type I constituent. In addition, there exists a single bubble S of type I constituent with mass $m \geq \overline{m_1}^s$ since Lemma 5.6.5. The energy will change be removing the the mass ε from the lobe of type I constituent and adding it to S as

$$\begin{aligned}\Delta = &\Big[\frac{\Gamma_{11}(x-\varepsilon)^2}{4\pi} + \frac{\Gamma_{12}(x-\varepsilon)m_2}{2\pi} + \frac{\Gamma_{22}m_2^2}{4\pi} + p(x-\varepsilon, m_2) + \frac{\Gamma_{11}(m+\varepsilon)^2}{4\pi} + 2\sqrt{\pi(m+\varepsilon)}\Big] \\ &- \Big[\frac{\Gamma_{11}x^2}{4\pi} + \frac{\Gamma_{12}xm_2}{2\pi} + \frac{\Gamma_{22}m_2^2}{4\pi} + p(x, m_2) + \frac{\Gamma_{11}m^2}{4\pi} + 2\sqrt{m\pi}\Big] \\ = &-\frac{\Gamma_{11}x\varepsilon}{2\pi} - \frac{\Gamma_{12}m_2\varepsilon}{2\pi} + p(x-\varepsilon, m_2) - p(x, m_2) + \frac{\Gamma_{11}m\varepsilon}{2\pi} + 2\sqrt{\pi(m+\varepsilon)} - 2\sqrt{\pi m} \\ = &\Big(\frac{\Gamma_{11}m}{2\pi} + \sqrt{\frac{\pi}{m}} - \frac{\Gamma_{11}x + \Gamma_{12}m_2}{2\pi} - \frac{1}{r_1}\Big)\varepsilon + O(\varepsilon^2),\end{aligned}$$

where r_1 represents the radius of the lobe of mass x. Besides, the following holds since the minimality of the minimizing configuration of (5.3.7)

$$0 \leq \frac{\Gamma_{11}m}{2\pi} + \sqrt{\frac{\pi}{m}} - \frac{\Gamma_{11}x + \Gamma_{12}m_2}{2\pi} - \frac{1}{r_1} \leq \frac{\Gamma_{11M_1^*}}{2\pi} + \sqrt{\frac{\pi}{\overline{m_1}^s}} - \sqrt{\frac{\pi}{3x}},$$

where M_1^* and $\overline{m_1}^s$ are only depended on Γ_{11}. Thus, the lower bound of x is proved. The proof for any lobe of type II constituent is the same.

Furthermore, consider a double bubble with lobes of i-th constituent have masses $x_i, i =$

1, 2. The energy will be changed by splitting it into two single bubbles with masses x_1 and x_2 by

$$\begin{aligned}\Delta &= \sum_{i=1}^{2} \frac{\Gamma_{ii} x_i^2}{4\pi} + 2\sqrt{\pi}(\sqrt{x_1} + \sqrt{x_2}) - \Big[\frac{\Gamma_{12} x_1 x_2}{2\pi} + \sum_{i=1}^{2} \frac{\Gamma_{ii} x_i^2}{4\pi} + p(x_1, x_2)\Big] \\ &\leq 2\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*}) - \frac{\Gamma_{12} \overline{m_1}^d \overline{m_2}^d}{2\pi}.\end{aligned}$$

The following holds since the minimality of the minimizing configuration of (5.3.7)

$$0 \leq 2\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*}) - \frac{\Gamma_{12} \overline{m_1}^d \overline{m_2}^d}{2\pi}.$$

Define

$$\Gamma_{12}^* := \frac{4\pi\sqrt{\pi}(\sqrt{M_1^*} + \sqrt{M_2^*})}{\overline{m_1}^d \overline{m_2}^d}.$$

In addition, since the M_i^* and $\overline{m_i}^d$ are only depended on $\Gamma_{ii}, i = 1, 2$ for all sufficiently large Γ_{12}. Thus, $\Gamma_{12} > \Gamma_{12}^*$ and there is not any double bubble.

□

Proof of Theorem 5.4.2 (One Double Bubble)

Assume there are two double bubbles that each lobe of i-th constituent has mass less than m_i^*. However, since Lemma 5.6.3, this situation is prohibited. Therefore, there exist at most one double bubble.

Assume there are two single bubbles of type I constituent with masses m_1^1 and m_1^2 respectively. The following holds

$$\begin{aligned}&[e_0(m_1^1 - \varepsilon, 0) + e_0(m_1^2 + \varepsilon, 0)] - [e_0(m_1^1, 0) + e_0(m_1^2, 0)] \\ =&\Big[\frac{\partial}{\partial m_1} e_0(m_1^2, 0) - \frac{\partial}{\partial m_1} e_0(m_1^1, 0)\Big]\varepsilon \\ &+ \frac{1}{2}\Big[\frac{\partial^2}{(\partial m_1)^2} e_0(m_1^1, 0) + \frac{\partial^2}{(\partial m_1)^2} e_0(m_1^2, 0)\Big]\varepsilon^2 + O(\varepsilon^3).\end{aligned}$$

Besides, since the requirements of the minimality of the minimizing configuration of (5.3.7),

$$\frac{\partial}{\partial m_1} e_0(m_1^2, 0) = \frac{\partial}{\partial m_1} e_0(m_1^1, 0) = 0.$$

Since the proof of Lemma 5.6.7 where $m_1^1, m_1^2 < M_1 < \min\{m_1^*, \pi\Gamma_{11}^{-2/3}\}$, the following holds

$$\frac{\partial^2}{(\partial m_1)^2} e_0(m_1^1, 0) < 0, \quad \frac{\partial^2}{(\partial m_1)^2} e_0(m_1^2, 0) < 0.$$

However, it is prohibited by the minimality of the minimizing configuration of (5.3.7). Therefore, there exist at most one single bubble of each constituent.

Assume there is a single bubble with mass m of type I constituent without loss of generality. Also, assume there is double bubble with lobes of masses m_1 and m_2. Then, the following holds

$$[e_0(m - \varepsilon, 0) + e_0(m_1 + \varepsilon, m_2)] - [e_0(m, 0) + e_0(m_1, m_2)]$$
$$\Big[\frac{\partial}{\partial m_1} e_0(m_1, m_2) - \frac{\partial}{\partial m_1} e_0(m, 0)\Big]\varepsilon + \frac{1}{2}\Big[\frac{\partial^2}{(\partial m_1)^2} e_0(m, 0) + \frac{\partial^2}{(\partial m_1)^2} e_0(m_1, m_2)\Big]\varepsilon^2 + O(\varepsilon^3).$$

Besides, since the requirements of the minimality of the minimizing configuration of (5.3.7), the following holds

$$\frac{\partial}{\partial m_1} e_0(m_1, m_2) = \frac{\partial}{\partial m_1} e_0(m, 0) = 0.$$

Since the combining of the Lemma 5.6.3 and the proof of Lemma 5.6.7 where $m_1^1, m_1^2 < M_1 < \min\{m_1^*, \pi\Gamma_{11}^{-2/3}\}$, the following holds

$$\frac{\partial^2}{(\partial m_1)^2} e_0(m, 0), \quad \frac{\partial^2}{(\partial m_1)^2} e_0(m_1, m_2).$$

However, it is prohibited by the minimality of the minimizing configuration of (5.3.7). Therefore, there exist no coexistence.

As a result, with the choice of Γ_{12}, there exists

$$\frac{\Gamma_{12}}{2\pi} M_1 M_2 + p(M_1, M_2) < 2\sqrt{\pi}(\sqrt{M_1} + \sqrt{M_2})$$

by comparing the case of one double bubble with lobes of masses M_1 and M_2 against the case of two single bubbles of different constituents with masses M_1 and M_2.

$\square$

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