AMENABLE SEMIGROUPS AND ERGODICITY

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ABSTRACT

For each μ-measure preserving map ϕ from a measure space (X,A,μ) into itself, the operator T defined on the Hilbert space $L_2(X,A,\mu)$ by

$$Tf(x) = f(\phi(x)),$$

for each f in $L_2(X,A,\mu)$ and x in X, is a unitary operator. The mean ergodic theorem of von Neumann asserts that the arithmetic means $T_n = \frac{1}{n}\sum_{i=1}^{n} T^i$ of the iterates $\{T^i\}_{i=1}^{\infty}$ converges strongly in $L_2(X,A,\mu)$. This was extended to $L_p(X,A,\mu)$, $1 \le p \le \infty$, by Riesz. Then Yosida and Kakutani generalized the above results to Banach spaces. They proved that if T is a bounded linear operator on a Banach space then the arithmetic means $\{T_n(b)\}$, b in B, converges strongly to b_0 if (i) $\sup_n \|T^n\| < \infty$ and (ii) b_0 is a weak cluster point of $\{T_n(b)\}$.

Eberlein has defined a semigroup S of bounded linear operators on a Banach space B to be ergodic if there is a net $\{A_n\}$ of averages of S such that the (i) $\sup_n \|A_n\| < \infty$, and (ii) the nets $\{A_n(s-I)\}$ and $\{(s-I)A_n\}$ converge to 0 for each s in S. From this definition, one can show that if b_0 is weak cluster point of $\{A_n(b)\}$, b in B, then $\{A_n(b)\}$ converges to b_0. Then, with Eberlein's definition of ergodicity, one can paraphrase the mean ergodic theorems of von Neumann, Riesz, Yosida and Kakutani as an assertion that the uniformly bounded cyclic semigroups generated by a bounded linear operator on a

Banach space is ergodic. One of the prime interest of this thesis is to bring in a result of M. M. Day which characterizes those semigroups which are ergodic when they are represented as a uniformly bounded linear operators from a Banach space into itself. These turn out to be the class of all amenable semigroups, i.e. those semigroups S which have a non-negative translation invariant linear functional of norm one on the Banach space of all bounded real-valued functions on S. Since the class of amenable semigroups includes the class of all Abelian semigroups, the theorems of von Neumann, Riesz, Yosida and Kakutani follow from Day's result.

A great portion of this thesis is devoted to the study of these amenable semigroups. Results by various authors on the characterizations and combinatorial properties of these amenable semigroups are given.

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INTRODUCTION

Let (X,A,μ) be a σ-finite measure space and ϕ be a μ-measure preserving bijection from X onto X. The mapping T from the Hilbert space $L_2(X,A,\mu)$ into itself defined by

$$(1) \qquad Tf(x) = f(\phi(x)),$$

for each f in $L_2(X,A,\mu)$ and each x in X, is a unitary operator. The mean ergodic theorem of J. von Neumann stated that, for each f in $L_2(X,A,\mu)$, the arithematic means $T_n f = (n+1)^{-1} \sum_{i=0}^{n} T^i f$ converges strongly in $L_2(X,A,\mu)$. His proof was based on the spectral theory of unitary operators on Hilbert spaces. It was then observed by F. Riesz that if T is defined on $L_p(X,A,\mu)$, as in (1), where $1 \leq p < \infty$, then $\{T_n f\}$ converges strongly on $L_p(X,A,\mu)$, for each f in $L_p(X,A,\mu)$. (Note that, for the case $p = 1$, the assumption that $\mu(X) < \infty$ is needed.) At the same time, Yosida [30], Yosida and Kakutani [31] proved, independently from F. Riesz, that if T is a bounded linear operator from a Banach space B into itself such that $\sup_i \| T^i \| < \infty$ and, for b in B, the arithematic means $T_n b = (n+1)^{-1} \sum_{i=0}^{n} T^i b$ has a subsequence converges weakly to b_0, for some b_0 in B, then $\{T_n b\}$ converges strongly to b_0 in B. Their result thus subsumed Riesz's result. If we consider $n \to T^n$ as a representation of the additive semigroup of all non-negative integers as bounded linear operators from B into itself, then the mean ergodic theorem may be considered as a result concerning the strong convergence of the means of this representation. This lead to a more general formulation for semigroup of operators by Alaoglu and Birkhoff [1]. It was Eberlein [8] who observed that ergodicity of a semigroup S of bounded linear operators from a Banach space B into itself really depended on the existence of a net $\{A_n\}$ of averages of S such that $\{A_n\}$ is uniformly bounded and $\lim_n A_n(s-I) = 0 = \lim_n (s-I)A_n$, for each s in S. Notice that the convergence of the nets $\{A_n(s-I)\}$ and $\{(s-I)A_n\}$ in different topologies gives rise to different strength of ergodicity. Say S is weakly, strongly and uniformly ergodic if $\{A_n(s-I)\}$ and $\{(s-I)A_n\}$ converge, for each s in S, to 0 in the weak, strong and uniform operator topology of the space of all bounded linear operators from B into B, respectively. One of the purposes of this thesis is to bring in a result of M. M. Day which characterizes those semigroups which are ergodic (weakly, strongly, and uniformly) when represented (or anti-represented) as uniformly bounded linear operators from a Banach space B into itself. These are precisely the so called amenable semigroups, i.e. those semigroup S in which there is a non-negative linear functional μ of norm one on the space of all bounded real-valued functions on S such that μ is invariant under left and right translations. In such a case, the various strengths of ergodicity are equivalent. The second purpose of this thesis is devoted to the studies of these amenable semigroups. Various characterizations, combinatorial properties, and examples of amenable semigroups are given.

The organization of this thesis is as follows: Chapter I §1, presents some basic concepts of functional analysis which we will use throughout this thesis. Then, in §2, we list some properties of the two function spaces, m(S) and $\ell_1(S)$ of a non-void set S, which we will encounter frequently in the subsequent chapters.

The second chapter is devoted to a survey of some results on amenable semigroups. Definitions of means, invariant means and their properties are given in §1. In §2, we introduce the Arens product on $m(S)^*$ which renders $m(S)^*$ a Banach algebra; and use this to facilitate the study of invariant means. In §3, we give various characterizations of amenable semigroups. Combinatorial properties and examples of amenable semigroups are given in §4.

In the final chapter, we bring in M. M. Day's result, which shows that the class of all amenable semigroups is exactly those that are ergodic when represented (or anti-represented) as uniformly bounded linear operators from a Banach space into itself.

CHAPTER I PRELIMINARIES

In this chapter we introduce some results on topological vector spaces on which the proofs in subsequent chapters are based. We assume that the basic concepts in general topology are familar to the reader. The standard reference for these concepts is [20]. Also, since the propositions given in this chapter are well-known, we will not bring in all the proofs. Nevertheless, for each proposition or theorem we state, at least one reference will be given. The standard references for results in topological vector space are [7, 17, 18, 21, 28].

§1. Topological Vector Spaces.

First, we note that all topologies we consider throughout this section are Hausdorff.

1.1. Definition. A topological vector space (E,τ) is a vector space E (real or complex) together with a topology τ such that the mappings $(x+y) \to x+y$ and $(\alpha,x) \to \alpha x$ are continuous.

It can be proved directly from the definition that, for a topological vector space (E,τ), the maps $y \to a+y$ for each fixed a in E and $y \to \alpha y$ for each fixed scalar α, $\alpha \neq 0$, are homeomorphisms from E onto itself. Hence, the neighborhood system of the origin determines the whole topology.

We say a subset A of a vector space is convex if $\lambda x+(1-\lambda)y$ is in A whenever x and y are in A and

$0 \leq \lambda \leq 1$. We are interested in those topological vector spaces which have a base of neighborhoods of the origin consisting of convex sets. Such spaces are called locally convex spaces, or simply convex space.

A non-negative real-valued function p on a vector sapce E is called a semi-norm if $p(x+y) \leq p(x) + p(y)$ and $p(\alpha x) = |\alpha| p(x)$, for all x and y in E and all scalar α.

1.2. Proposition. *Let* (E,τ) *be a locally convex space. Then* τ *is generated by a family of semi-norms.*

Conversely, if $\{p_i\}$ $i \in I$ *is a family of semi-norms on a vector space* E*, the weakest topology that makes each* p_i*,* i *in* I*, continuous is a locally convex topology for* E*. Moreover, the topology generated by* $\{p_i\}$ $i \in I$ *is Hausdorff if and only if there is, for each* $x \neq 0$*, some* p_i *such that* $p_i(x) \neq 0$.

See [28, Theorem 3 and Proposition 8, p. 15] for a proof.

In view of the previous proposition, every family of semi-norms on a vector space determines a locally convex topology on the space. Important examples of locally convex space are those generated by a single semi-norm p (we usually denote $p(x)$ by $\|x\|$) with the additional property that $\|x\| > 0$ if $x \neq 0$. Such a semi-norm $\| \ \|$ is called a norm and the locally convex space generated by $\| \ \|$ is called a normed vector space, or normed space. If a normed space is complete with respect to the metric generated by the norm, then it is called a Banach space.

A linear transformation $T:E \to F$ from a normed space E into another normed space F is bounded if

$$\|T\| = \inf\{\|Tx\|:x \text{ in } E \text{ and } \|x\| \leq 1\}$$

exists and is finite. It is well-known that T is bounded if and only if it is continuous. (See [7, Lemma 4, p. 59].) Furthermore, the real-valued function $\|T\|$ defined as above on the vector space of all bounded linear transformations from E into F is a norm.

1.3. Proposition. *Let* E *and* F *be normed spaces and* $B(E,F)$ *be the normed space of all bounded linear transformations from* E *into* F. *If* F *is complete, then* $B(E,F)$ *is also complete.*

See [7, Lemma 8, p. 61] for a proof.

The importance of locally convex spaces is that they have sufficiently many continuous linear functionals to seperate points. This is a consequence of the Hahn-Banach Theorem.

1.4. Theorem. (Hahn-Banach) *Let* E *be a real vector space and* p *be a sublinear functional, i.e.* p *is real-valued with*

$$p(x+y) \leq p(x) + p(y) \quad \textit{and} \quad p(\alpha x) = \alpha p(x)$$

for all x *and* y *in* E *and every* $\alpha \geq 0$. *For every real* *linear functional* f *on a vector subspace* H *of* E *with* $f(x) \leq p(x)$, *for all* x *in* H, *there is a real linear functional* f_0 *on* E *such that*

$$f_0(x) = f(x) \quad \textit{and} \quad f(y) \leq p(y)$$

for each x *in* H *and each* y *in* E.

See [7, Theorem 10, p. 62] for a proof.

In this thesis, we are interested in the following forms of the Hahn-Banach Theorem.

1.5. <u>Corollary</u>. *Let* E *be a real normed vector space and* f *be any real continuous linear functional on a vector subspace* H *of* E. *Then there is a real linear functional* f_0 *on* E *such that* $f_0(x) = f(x)$, *for each* x *in* H, *and* $\|f_0\| = \|f\|$.

See [7, Theorem 11, p. 62] for a proof.

1.6. <u>Corollary</u>. *Let* (E,τ) *be a real locally convex space and let* A *and* B *be disjoint closed convex sets in* E. *If* A *is compact, then there is a continuous linear functional* f *on* E *and constants* c *and* ε, $\varepsilon > 0$, *such that*

$$f(a) \leq c - \varepsilon < c \leq f(b)$$

for every a *in* A *and every* b *in* B.

See [7, Theorem 10, p. 417] for a proof.

For each topological vector space (E,τ) we denote by E^*, the dual space of E, the vector space of all continuous linear functionals on E. For each f in E^*, define a semi-norm p_f on E by

$$p_f(x) = |f(x)|$$

for all x in E. Then the family $\{p_f : f \text{ in } E^*\}$ of semi-norms determines a locally convex topology on E, namely the weakest topology that makes each p_f, f in E^*, continuous. If (E,τ) is a locally convex space, then, by Corollary 1.6 and Proposition 1.2, this topology is Hausdorff. We call this topology the weak topology, or simply ω-topology, on E induced by E^* and denote it by $\sigma(E,E^*)$. In this topology, a net $\{x_n\}$ in E converges to x in $\sigma(E,E^*)$ if and only if $\lim_n f(x_n) = f(x)$, for each f in E^*. In this case, we say $\{x_n\}$ converges weakly to x and write $\omega\text{-}\lim_n x_n = x$.

In general, $\sigma(E,E^*)$ is weaker than the original topology τ on E. However, for a convex set A in E, the closure of A in τ is the same as the closure of A in $\sigma(E,E^*)$. For future references, we put this down formally in the following proposition in a more general form.

1.7. Proposition. *Let* E *be a vector space. Suppose that* E *is given two locally convex topologies* τ_1 *and* τ_2 *such that the dual spaces of* E *with respect to these topologies are the same.*

Then, a convex set in E *is closed in* τ_1 *if and only if it is closed in* τ_2.

See [7, Corollary 14, p. 418] for a proof.

Let $\{E_i\}$ be a family of vector spaces. Then the full direct product $\prod_{i\varepsilon I} E_i$ of $\{E_i\}$ forms a vector space under the operations $((x_i),(y_i)) \to (x_i+y_i)$ and $(\alpha,(x_i)) \to (\alpha x_i)$.

1.8. Proposition. *Let* $\{(E_i,\tau_i)\}_{i\varepsilon I}$ *be a family of locally convex spaces. Then the full direct product* $E = \prod_{i\varepsilon I} E_i$ *together with the product topology* τ *of the topologies* τ_i *forms a locally convex space.*

Moreover, the weak topology $\sigma(E,E^*)$ *on* E *induced by* E^* *is exactly the product topology of the topologies* $\sigma(E_i,E_i^*)$.

See [21, (17.13), p. 160] for a proof.

Let (E,τ) be a locally convex space. For each x in E, we define a semi-norm P_x on E^* by

$$P_x(f) = |f(x)|$$

for all f in E^*. If $f \neq 0$ in E^*, then there is an x in E such that $P_x(f) = |f(x)| > 0$. Hence, the family $\{P_x : x \text{ in } E\}$ induces a Hausdorff locally convex topology on E^*, called the weak*-topology, or simply ω^*-topology, on E^*, and is denoted by $\sigma(E^*,E)$. In this topology, a net $\{f_n\}$ in E^* converges to t in $\sigma(E^*,E)$ if and only if $\lim_n f_n(x) = f(x)$, for each x in E. In such case, we write $\omega^*\text{-}\lim_n f_n = f$.

Let E be a normed space. By Proposition 1.3, we know that E^* is also a normed space. Let E^{**} denote the continuous dual of E^* with respect to the norm-topology on E^*. Sometimes, we call E^{**} the second dual of E. Consequently, together with the norm-topology on E^*, the space E^* has the ω-topology $\sigma(E^*,E^{**})$ induced by E^{**} and the ω^*-topology $\sigma(E^*,E)$ induced by E. In general, $\sigma(E^*,E)$ is weaker than $\sigma(E^*,E^{**})$ while $\sigma(E^*,E^{**})$ is weaker than the norm-topology on E^*.

Let (E,τ) and (F,μ) be locally convex spaces and $T:E \to F$ be continuous and linear. For each f in F^*, define T^*f in E^* by

$$T^*f(x) = f(Tx)$$

for all x in E. Since T and f are continous and linear, T^*f is the composition of two continuous linear maps. Hence, T^*f is in E^*. Thus, the linear transformation $T^*:f \to T^*f$ from F^* into E^* is well-defined. We call T^* the adjoint operator of T.

1.9. <u>Proposition</u>. *Every adjoint operator is ω^*-ω^*-continuous.*

See [7, Lemma 3, p. 478] for a proof.

Let E be a normed space. Proposition 1.3 shows that

E^* is a Banach space. It is known that the closed unit ball $\{x^* \text{ in } E^* : \|x^*\| \leq 1\}$ in E^* is not always compact in the norm topology.

1.10. Theorem. (Alaoglu) *Let* B^* *be the continuous dual of a Banach space* B. *Then the closed unit ball is compact in the* ω^*-*topology of* B^*.

See [7, Theorem 2, p. 424] for a proof.

Let E be a normed space and E^* and E^{**} be the dual and the second dual of E, respectively. For each x in E, define a map Q from E into E^{**} by

$$Qx(f) = f(x),$$

for x in E and f in E^*. Since Qx depends linearly and continuously on f, Qx is in E^{**}. By the Hahn-Banach Theorem, Q is one-to-one and preserves norm. Hence, Q is an embedding of E into E^{**}. This map Q is called the natural embedding of E into E^{**}. We summarize the above in the following proposition:

1.11. Proposition. *The mapping* $Q: E \to E^{**}$ *as defined above from a normed space* E *into its second dual* E^{**} *is an isometric isomorphism from* E *into* E^{**}. *That is,* Q *is linear, one-to-one and* $\|Qx\| = \|x\|$, *for all* x *in* E.

See [7, Theorem 19, p. 66] for a proof.

1.12. Proposition. *Let* Q *be the natural embedding of a Banach space* B *into* B**. *Then* QB *is dense in* B** *with respect to the* ω^**-topology* $\sigma(B^{**},B^*)$ *of* B**.

For a proof, see [7, Corollary 6, p. 425].

§2. Two Special Banach Spaces.

We give in this section some properties of two special Banach spaces which we will encounter throughout this thesis.

Let S be a non-void set. We denote by m(S) the real Banach space of all bounded real-valued function on S with the norm defined by

(2.0.1) $$\|f\| = \sup\{|f(x)|:s \text{ in } S\},$$

for each f in m(S). Let $\ell_1(S)$ denote the real Banach space of all real-valued functions ϕ on S, such that $\sum_{s\varepsilon S}|\phi(s)|$ exists and is finite, with the norm defined by

(2.0.2) $$\|\phi\| = \sum_{s\varepsilon S}|\phi(s)|,$$

for each ϕ in $\ell_1(S)$. Here, the sum $\sum_{s\varepsilon S}|\phi(s)|$ means $\lim_{\sigma\varepsilon\Sigma}\sum_{s\varepsilon\sigma}|\phi(s)|$, where the limit is taken with respect to the directed set Σ of all finite subsets of S ordered by set inclusion. It is well-known that if ϕ is in $\ell_1(S)$, then the support of ϕ, $\{s \text{ in } S:\phi(s) \neq 0\}$, is countable. (See [15, Theorem 1, p. 19].)

In the following, we give the notations of some special elements in m(S) and $\ell_1(S)$ which will be used throughout this thesis.

2.1. Notations. Let S be a non-void set and let m(S) and $\ell_1(S)$ be the Banach spaces defined as above.

(2.1.1) For each subset A in S, define 1_A in m(S) by $1_A(s) = 1$ if s in A and $1_A(s) = 0$ otherwise. In particular, we write $1 = 1_S$ and $1_s = 1_{\{s\}}$, for each s in S.

(2.1.2) Suppose f and g are in m(S). We write $f \geq g$ if $f(s) \geq g(s)$, for each s in S. In particular, when $f \geq 0$, where 0 is the zero function in m(S), we call f is non-negative.

2.2. Theorem. *The mapping* $J:f \to Jf$ *from* m(S) *into* $\ell_1(S)^*$ *defined by*

$$Jf(\phi) = \textstyle\sum_{s\in S} f(s)\phi(s),$$

for f *in* m(S) *and* ϕ *in* $\ell_1(S)$ *is an isometric isomorphism from* m(S) *onto* $\ell_1(S)^*$. *That is:*

(2.2.1) J *is onto, one-to-one and linear;*

(2.2.2) $$\|Jf\| = \|f\|,$$

for each f *in* m(S).

See [18, Theorem 20.20, p. 353] for a proof.

In view of Theorem 2.2 and Proposition 1.1, the next corollary is evident.

2.3 Corollary. *The mapping* $Q:\ell_1(S) \to m(S)^*$ *defined by, for each* ϕ *in* $\ell_1(S)$ *and each* f *in* m(S),

$$Q(\phi)(f) = \textstyle\sum_{s\epsilon S} f(s)\phi(s),$$

is a natural embedding of $\ell_1(S)$ *into* $m(S)^*$.

Another topology on the function space m(S) which will be useful in the later chapters is the pointwise topology. The pointwise topology on m(S) is the relative toplogy of the product topology on the product space $\prod_{s\epsilon S} R_s$, where for each s, R_s is the reals with the usual topology. In this topology, a net $\{f_n\}$ in m(S) converges to f in m(S) if and only if $\lim_n f_n(s) = f(s)$, for each s in S; and we say $\{f_n\}$ converges pointwise to f. Moreover, this topology is Hausdorff and is weaker than the ω^*-topology on m(S), since the evaluation map is a linear functional on m(S). However, they agree on any norm-bounded set in m(S). This follows easily from an application of Theorem 1.10 and [20, Theorem 2, p. 220]. We put this down formally in the following proposition.

2.4. Proposition. *Let* A *be a norm-bounded and* ω^**-closed set in* m(S). *Then the* ω^**-topology and the pointwise topology of* m(S) *agree on* A.

CHAPTER II AMENABLE SEMIGROUPS

The study of amenable semigroups began when Hausdorff [16] showed that no means exist on the space of all bounded real-valued functions on the surface of the 3-sphere which is invariant under rotation. Then, Banach [4] showed that there is a mean on the space of all bounded real-valued functions on the positive integer which is invariant under translations. J. von Neumann explained that the reason of the failure of the first case is the excessive non-commutativity of the rotation group of the 3-sphere, since there are plentiful of non-Abelian subgroups of the rotation group. Then M. M. Day [5] brought the subject to attention in his study of the ergodicity of bounded operator semigroups.

The main purpose of this chapter is to display the properties of amenable semigroups. In the first section of this chapter, we give the definitions and basic properties of means and invariant means. In §2, we show how an associative multiplication can be defined on the Banach space $m(S)^*$ so that $m(S)^*$ forms a Banach algebra. Then, we use this to facilitate our studies of the set of invariant means. The third section is devoted to the various characterizations of amenable semigroups. Finally, in §4, we bring in some combinatorial properties of amenable semigroups.

1. Means and Invariant Means.

This section is devoted to the properties of means on m(S); and especially those that are invariant under translations. First, we give a definition of a mean on m(S).

Let S be a fixed non-void set.

1.1. <u>Definition</u>. An element μ in $m(S)^*$ is called a mean on m(S) if

(1.1.1) $$\inf\{f(s): s \text{ in } S\} \leq \mu(f) \leq \sup\{f(s): s \text{ in } S\},$$

for each f in m(S).

The following proposition gives some equivalent conditions for a linear functional on m(S) to be a mean.

1.2. <u>Proposition</u>. *If* μ *is a mean on* m(S), *then it satisfies the following properties:*

(1.2.1) $\mu(f) \geq 0$ if $f \geq 0$;

(1.2.2) $\mu(1) = 1$;

(1.2.3) $\|\mu\| = 1$.

Conversely, if μ *in* $m(S)^*$ *satisfies any two of the conditions* (1.2.1), (1.2.2) *and* (1.2.3), *then* μ *is a mean.*

<u>Proof</u>. Suppose μ is a mean on m(S). Then (1.2.1) and (1.2.2) follows from the following inequalities:

$$\mu(f) \geq \inf\{f(s): s \text{ in } S\} \geq 0,$$

if $f \geq 0$; and

$$1 = \inf\{1(s):s \text{ in } S\} \leq \mu(1) \leq \sup\{1(s):s \text{ in } S\} = 1.$$

Since (1.2.2) holds, it follows that

(1.2.4) $$1 = |\mu(1)| \leq \|\mu\| \, \|1\| = \|\mu\|.$$

For any f in $m(S)$, it follows from the definition of a mean that

$$-\sup\{-f(s):s \text{ in } S\} \leq \mu(f) \leq \sup\{f(s):s \text{ in } S\}.$$

Thus, $|\mu(f)| \leq \sup\{|f(s)|:s \text{ in } S\} = \|f\|$. Consequently, $\|\mu\| \leq 1$. Combining this with (1.2.4), we have $\|\mu\| = 1$.

Conversely, suppose μ in $m(S)^*$ satisfies (1.2.1) and (1.2.2). From (1.2.1), we have $\mu(f) \leq \mu(g)$ whenever $f \leq g$. For each f in $m(S)$, let $\alpha = \inf\{f(s):s \text{ in } S\}$ and $\beta = \sup\{f(s):s \text{ in } S\}$. Since $\alpha 1 \leq f \leq \beta 1$, it follows from (1.2.2) that

$$\alpha = \alpha\mu(1) \leq \mu(f) \leq \beta\mu(1) = \beta.$$

Hence, μ is a mean on $m(S)$.

Suppose now that μ satisfies (1.2.1) and (1.2.3). It is sufficient to show that (1.2.2) holds. Since $1 \geq 0$, by (1.2.1) and (1.2.3), we have

$$0 \leq \mu(1) = |\mu(1)| \leq \|\mu\| \, \|1\| = 1.$$

On the other hand, by (1.2.1), it follows that

$$\mu(f) \leq \sup\{f(s):s \text{ in } S\},$$

for each f in $m(S)$. In particular,

$$\mu(-1) \leq \sup\{-1(s):s \text{ in } S\} = -1.$$

Hence, $\mu(1) = -\mu(-1) \geq -1$ and (1.2.2) holds.

Finally, suppose μ satisfies (1.2.2) and (1.2.3). To prove that μ is a mean it is sufficient to show that μ satisfies (1.2.1). Let $f \geq 0$ in $m(S)$ be arbitrary and let $\beta = \sup\{f(s):s \text{ in } S\} = \|f\|$ and $\alpha = \inf\{f(s):s \text{ in } S\}$. Then, by (1.2.3)

$$\begin{aligned} \beta-\mu(f) &= \mu(\beta 1-f) \\ &\leq \|\mu\| \, \|\beta 1-f\| \\ &= \|\beta 1-f\| \\ &= \sup\{\beta-f(s):s \text{ in } S\} \\ &= \beta + \sup\{-f(s):s \text{ in } S\} \\ &= \beta-\alpha \end{aligned}$$

Thus, $\mu(f) \geq \alpha \geq 0$ whenever $f \geq 0$.

1.3. Definition. Let ϕ be in $\ell_1(S)$. Then, ϕ is called a countable mean on S if

(1.3.1) $\qquad \phi(s) \geq 0$, for all s in S;

(1.3.2) $\qquad \sum_{s\varepsilon S}\phi(s) = 1$.

A countable mean ϕ on S is called a finite mean on S if its support, $\{s \text{ in } S:\phi(s) \neq 0\}$, is finite.

1.4. Proposition. *If* ϕ *in* $\ell_1(S)$ *is a countable, or finite* *mean on* S, *then* $Q\phi$, *where* Q *is the natural embedding of* $\ell_1(S)$ *into* $m(S)^*$, *is a mean on* $m(S)$.

Proof. If ϕ is a countable, or finite, mean on S, then since $\phi(s) \geq 0$, for all s in S, and $\sum_{s\in S}\phi(s) = 1$, we have $\|\phi\| = 1$. Since Q is isometric, $\|Q\phi\| = \|\phi\| = 1$. If $f \geq 0$, then

$$Q\phi(f) = \sum_{s\in S}\phi(s)f(s) \geq 0.$$

By Proposition 1.2, $Q\phi$ is a mean on $m(S)$.

1.5. Remark. If ϕ is a countable mean, or a finite mean, on S, then we call $Q\phi$ a countable mean, or finite mean, on $m(S)$. Let Φ denote the set of all finite means on S. Then $Q\Phi$ is the set of all finite means on $m(S)$. Since Q is an isometric isomorphism between $\ell_1(S)$ and $m(S)^*$, we use Φ, instead of $Q\Phi$, for the set of all finite means on $m(S)$ when no confusion arises. Since, for each s in S, 1_s is a finite mean on S, Φ is non-void. Moreover, if ϕ and σ are finite means and $0 \leq \alpha \leq 1$, then

$$\begin{aligned}
&\sum_{s\in S}[\alpha\phi(s)+(1-\alpha)\sigma(s)] \\
&\quad = \sum_{s\in S}\alpha\phi(s)+\sum_{s\in S}(1-\alpha)\sigma(s) \\
&\quad = \alpha \sum_{s\in S}\phi(s)+(1-\alpha) \sum_{s\in S}\sigma(s) \\
&\quad = \alpha+(1-\alpha) \\
&\quad = 1
\end{aligned}$$

and $\alpha\phi(s)+(1-\alpha)\sigma(s) \geq 0$, for all s in S. Also, the support of $\alpha\phi+(1-\alpha)\sigma$ is finite. Hence, $\alpha\phi+(1-\alpha)\sigma$ is a finite mean and Φ is convex.

1.6. Theorem. *Let* M *be the set of all means on* m(S). *Then* M *is non-void, convex and* ω^**-compact.*

Proof. By Remark 1.5, the set Φ of all finite means on S is not empty. Also, by Proposition 1.4, $\Phi \subseteq M$. Hence, M is non-void.

To prove that M is convex, let μ and λ be means on $m(S)$. If $0 \leq a \leq 1$, then

$$\begin{aligned}[a\mu+(1-a)\lambda](1) &= a\mu(1) + (1-a)\lambda(1) \\ &= a+(1-a) \\ &= 1\end{aligned}$$

and, if $f \geq 0$,

$$[a\mu+(1-a)\lambda](f) = a\mu(f)+(1-a)\lambda(f) \geq 0.$$

This shows that $a\mu+(1-a)\lambda$ is a mean and hence M is convex.

Since the unit ball of $m(S)^*$ is ω^*-compact (Theorem I.1.10), we show that M is ω^*-compact by proving that M is ω^*-closed in the unit ball of $m(S)^*$. Let $\{\mu_n\}$ be a net in M such that $\omega^*\text{-}\lim_n \mu_n = \mu$. Then

$$\mu(1) = \lim_n \mu_n(1)$$

$$= \lim_n 1$$

$$= 1.$$

Also, for each n, $\mu_n(f) \geq 0$ whenever $f \geq 0$. Hence $\mu(f) = \lim_n \mu_n(f) \geq 0$ if $f \geq 0$. Evidently, μ is a mean and it follows that M is ω^*-closed in the unit ball.

1.7. Proposition. *The set* Φ *of all finite means on* $m(S)$ *is* ω^**-dense in* M.

Proof. Suppose on the contrary that there is a μ in M such that μ is not in the ω^*-closure of Φ. Since the ω^*-closure of Φ is also convex, by Corollary I.1.6, there is a ω^*-continuous linear functional on $m(S)^*$ which seperates Φ and μ. However, the ω^*-continuous linear functionals on $m(S)^*$ are exactly $m(S)$. Consequently, there is a f in $m(S)$ and constants c and ε, $\varepsilon > 0$, such that

$$\sup\{\phi(f) : \phi \text{ in } \Phi\} \leq c-\varepsilon<c \leq \mu(f).$$

In particular,

$$\begin{aligned} \sup\{f(s) : s \text{ in } S\} &= \sup\{Q1_s(f) : s \text{ in } S\} \\ &\leq \sup\{\phi(f) : \phi \text{ in } \Phi\} \\ &\leq c - \varepsilon < c \leq \mu(f). \end{aligned}$$

This contradicts that μ is a mean. Thus, Φ is ω^*-dense in M.

From now on, let S always be a semigroup, i.e. a non-void set S together with an associative binary operation $(s,t) \to st$ on S. For each s in S, the semi-group structure induces two linear operators ℓ_s and γ_s on $m(S)$ into itself defined by, respectively,

$$\ell_s f(t) = f(st) \quad \text{and} \quad \gamma_s f(t) = f(ts),$$

for each f in $m(S)$ and t in S. Since $|\ell_s f(t)| \leq \|f\|$ and $|\gamma_s f(t)| \leq \|f\|$, for all t in S, $\ell_s f$ and $\gamma_s f$ are in $m(S)$. Also, it is easy to check that the mappings $\ell_s : f \to \ell_s f$ and $\gamma_s : f \to \gamma_s f$, for each s in S, are linear and bounded by 1. Furthermore, the adjoint operator ℓ_s^* and γ_s^* of ℓ_s and γ_s, for each s in S, is also bounded by 1. Also, it follows directly from the definition that $\ell_{st} = \ell_t \ell_s$, $\gamma_{st} = \gamma_s \gamma_t$, $\ell_{st}^* = \ell_s^* \ell_t^*$ and $\gamma_{st}^* = \gamma_t^* \gamma_s^*$ for every s and every t in S.

1.8. <u>Definition</u>. Let S be a semigroup. A mean μ on $m(S)$ is called left [right] invariant if $\mu(f) = \mu(\ell_s f)$ $[\mu(f) = \mu(\gamma_s f)]$, for every f in $m(S)$ and every s in S. An equivalent definition is that μ is a left [right] invariant mean if $\ell_s^* = \mu$ $[\gamma_s^* \mu = \mu]$, for each s in S. If μ is both left and right invariant, then it is called a two-sided invariant mean, or simply an invariant mean.

1.9. <u>Definition</u>. A semi-group S is called a left [right] amenable semigroup if the space m(S) admits a left [right] invariant mean. If m(S) admits an invariant mean, then S is called amenable.

Examples of amenable semigroups are Abelian semigroups, solvable groups and finite groups. The proofs of these facts will be given in §4 of this chapter.

Let μ be a left [right] invariant mean for the left [right] amenable semigroup S. By Proposition 1.8, there is a net $\{\phi_n\}$ of finite mean which converges to μ in the ω^*-topology of m(S). Since every adjoint operator is ω^*-ω^*-continuous (see Proposition I.1.9), for each s in S, we have

$$\omega^*\text{-}\lim_n \ell_s^*\phi_n = \ell_s^*\mu = \mu$$

$$[\omega^*\text{-}\lim_n \gamma_s^*\phi_n = \gamma_s^*\mu = \mu].$$

Consequently, we have

$$\omega^*\text{-}\lim_n \left(\ell_s^*\phi_n - \phi_n\right) = 0 \quad [\omega^*\text{-}\lim_n \left(\gamma_s^*\phi_n - \phi_n\right) = 0],$$

for every s in S. An immediate question is whether the existence of such a net which satisfies the above condition is sufficient for the amenability of S. First, we define the followings.

1.10. Definition. Let $\{\mu_n\}$ be a net of means on m(S). Then $\{\mu_n\}$ is called ω^*-convergent [norm-convergent] to left invariance if

$$\omega^*\text{-}\lim_n (e_s^*\mu_n - \mu_n) = 0 \quad [\lim_n \| e_s^*\mu_n - \mu_n \| = 0]$$

for each s in S.

1.11. <u>Proposition</u>. *Suppose that* $\{\mu_n\}$ *is a net of means on* $m(S)$ *such that* $\omega^*\text{-}\lim_n \mu_n = \mu$ $[n\text{-}\lim_n \mu_n = \mu]$. *Then* μ *is a left or right invariant mean if and only if* $\{\mu_n\}$ *is* ω^**-convergent [norm-convergent] to left or right invariance respectively.*

<u>Proof</u>. We prove only the case of left invariance. The necessity follows from the previous discussion. The sufficiency follows from the identities

$$\mu = \omega^*\text{-}\lim_n \mu_n = \omega^*\text{-}\lim_n \ell_s^*\mu_n = \ell_s^*\mu$$

$$[\mu = n\text{-}\lim_n \mu_n = n\text{-}\lim_n \ell_s^*\mu_n = \ell_s^*\mu],$$

for all s in S.

1.12. <u>Corollary</u>. *If* $\{\mu_n\}$ *is a net of means which is* ω^**-convergent to left [right] invariance then every* ω^**-cluster point of* $\{\mu_n\}$ *is a left [right] invariant mean.*

<u>Proof</u>. If μ is a ω^*-cluster point of $\{\mu_n\}$, then there is a subnet $\{\mu_k\}$ of $\{\mu_n\}$ such that $\omega^*\text{-}\lim_k \mu_k = \mu$. By Proposition 1.11, μ is a left [right] invariant mean.

1.13. <u>Corollary</u>. *A semigroup* S *is left [right] amenable if and only if* $m(S)$ *admits a net* $\{\phi_n\}$ *of finite means which is* ω^*-

convergent to left [right] invariance.

Proof. The Corollary follows from Theorem 1.6, Propositions 1.7, 1.11, and Corollary 1.12.

§2. The Arens Product.

In this section, we introduce the Arens product defined on $m(S)^*$ which makes the Banach space $m(S)^*$ into a Banach algebra. Moreover, under this product, the set of all means on $m(S)$ forms a semigroup. First, we give the definition of a Banach algebra.

2.1. Definition. A Banach algebra B is a Banach space together with a binary operation $(b_1,b_2) \to b_1 \cdot b_2$ which satisfies the following properties:

(2.1.1) $(b_1 \cdot b_2) \cdot b_3 = b_1 \cdot (b_2 \cdot b_3)$;

(2.1.2) $b_1 \cdot (b_2 + b_3) = b_1 \cdot b_2 + b_1 \cdot b_3$

and $(b_2 + b_3) \cdot b_1 = b_2 \cdot b_1 + b_3 \cdot b_1$;

(2.1.3) $\alpha(b_1 \cdot b_2) = (\alpha b_1) \cdot b_2 = b_1 \cdot (\alpha b_2)$;

(2.1.4) $\| b_1 \cdot b_2 \| \leq \| b_1 \| \, \| b_2 \|$,

for all b_i, $i = 1, 2, 3$, in B and all scalars α.

Let S be a semigroup and let ϕ and σ be two elements in $\ell_1(S)$. Define a function $\phi\sigma$ on S by

$$\phi\sigma(s) = \sum_{xy=s} \phi(x)\sigma(y),$$

for all s in S. Since

$$\sum_{s\in S}|\phi\sigma(s)| = \sum_{s\in S}|\sum_{xy=s}\phi(x)\sigma(y)|$$

$$\leq \sum_{s\in S}\sum_{xy=s}|\phi(x)||\sigma(y)|$$

$$= \sum_{x\in S}|\phi(x)| \quad \sum_{y\in S}|\sigma(y)|$$

$$= \|\phi\| \, \|\sigma\|,$$

$\phi\sigma$ is well defined and is in $\ell_1(S)$. Hence $(\phi,\sigma) \to \phi\sigma$ is a binary operation on $\ell_1(S)$.

2.2. Proposition. *The Banach space* $\ell_1(S)$ *under the binary operation* $(\phi,\sigma) \to \phi\sigma$ *defined above forms a Banach algebra. Furthermore, the mapping* $s \to 1_s$ *is a semigroup isomorphism from* S *into the multiplicative semigroup of the Banach algebra* $\ell_1(S)$. *That is,* $1_{st} = 1_s 1_t$, *for all* s *and* t *in* S.

Proof. Since $\|\phi\sigma\| = \sum_{s\in S}|\phi\sigma(s)| \leq \|\phi\| \, \|\sigma\|$, for any ϕ and α in $\ell_1(S)$, condition (2.1.4) holds. The properties (2.1.2) and (2.1.3) of Definition 2.1 follows directly from the distributive law and commutative law of the reals, respectively.

For the associativity, let ϕ,σ and ζ be in $\ell_1(S)$. For each s in S,

$$[(\phi\sigma)\zeta](s) = \sum_{s_1 s_2=s}\phi\sigma(s_1)\zeta(s_2)$$

$$= \sum_{s_1 s_2=s}[\sum_{t_1 t_3=s_1}\phi(t_1)\sigma(t_3)]\zeta(s_2)$$

$$= \sum_{(t_1 t_3) s_2 = s} [\phi(t_1)\sigma(t_3)]\zeta(s_2)$$

$$= \sum_{t_1 t_2 = s} \phi(t_1) [\sum_{t_3 s_2 = t_2} \sigma(t_3)\zeta(s_2)]$$

$$= \sum_{t_1 t_2 = s} \phi(t_1)\sigma\zeta(t_2)$$

$$= [\phi(\sigma\zeta)](s) .$$

Hence, $\ell_1(S)$ is a Banach algebra.

Finally, let s and t be in S. Then

$$1_s 1_t(x) = \sum_{x = x_1 x_2} 1_s(x_1) 1_t(x_2) .$$

Hence, $1_s 1_t(x) = 1$ if $x = st$ and $1_s 1_t(x) = 0$ otherwise. Thus, $1_s 1_t = 1_{st}$. This completes the proof.

In [2], Arens showed how an associative multiplication can be defined on the second conjugate B^{**} of a Banach algebra B. This multiplication makes B^{**} into a Banach algebra and extends the multiplication in B. In this thesis, we are interested in the special case when B is the Banach algebra $\ell_1(S)$.

Now, we follow Arens' procedure in defining a multiplication in $\ell_1(S)^{**}$. It consists of three steps:

(A) For each f in $\ell_1(S)^*$ and ϕ in $\ell_1(S)$, define a function $f * \phi$ in $\ell_1(S)^*$ by

$$f*\phi(\sigma) = f(\phi\sigma) ,$$

for all σ in $\ell_1(S)$.

(B) For each μ in $\ell_1(S)^{**}$ and f in $\ell_1(S)^*$, define a function $\mu * f$ in $\ell_1(S)^*$ by

$$\mu * f(\phi) = \mu(f * \phi),$$

for each ϕ in $\ell_1(S)$.

(C) For each pair λ and μ in $\ell_1(S)^{**}$, define $\lambda * \mu$ in $\ell_1(S)^{**}$ by

$$\lambda * \mu(f) = \lambda(\mu * f),$$

for all f in $\ell_1(S)^*$.

2.3. Remark. We remark that the above operations are well-defined. First, it follows easily from the fact that $\ell_1(S)$ is a Banach algebra, the function $f * \phi$ defined in (A) is in $\ell_1(S)^*$. Also, the following properties can be checked easily:

(2.3.1) $\qquad \| f * \phi \| \le \| f \| \, \| \phi \|$;

(2.3.2) $\qquad f * (\phi + \sigma) = f * \phi + f * \sigma$;

(2.3.3) $\qquad (f + g) * \phi = f * \phi + g * \phi$;

(2.3.4) $\qquad f * (\alpha\phi) = \alpha(f * \phi) = (\alpha f) * \phi$

for all f and g in $\ell_1(S)^*$, all ϕ and σ in $\ell_1(S)$ and all scalars α.

Using these properties, we can prove that the function defined in (B) is indeed in $\ell_1(S)^*$. Furthermore, the following properties follows immediately from the properties (2.3.1) to

(2.3.4):

(2.3.5) $\|\mu*f\| \le \|\mu\|\,\|f\|$;

(2.3.6) $\mu*(f+g) = \mu*f+\mu*g$;

(2.3.7) $\mu*(\alpha f) = \alpha(\mu*f) = (\alpha\mu)*f$,

for every μ in $\ell_1(S)^{**}$ and all f and g in $\ell_1(S)^*$ and all scalars α.

Finally, suppose that $\lambda*\mu$ is the function defined in (C). Then by (2.3.6) and (2.3.7), $\lambda*\mu$ is linear. By (2.3.5), we have

$$\begin{aligned} |\lambda*\mu(f)| &= |\lambda(\mu*f)| \\ &\le \|\lambda\|\,\|\mu*f\| \\ &\le \|\lambda\|\,\|\mu\|\,\|f\| . \end{aligned}$$

Hence, $\lambda*\mu$ is in $\ell_1(S)^{**}$ and

(2.3.8) $\|\lambda*\mu\| \le \|\lambda\|\,\|\mu\|$,

for all λ and μ in $\ell_1(S)^{**}$.

Therefore, the operation $*$ is well-defined. Hereafter, we called $*$ the Arens product.

2.4. Lemma. *Let* λ *and* μ *be in* $\ell_1(S)^{**}$ *and* f *in* $\ell_1(S)^*$. *Then*

(2.4.1) $(\lambda*\mu)*f = \lambda*(\mu*f)$.

Proof. For each f in $\ell_1(S)^*$ and all ϕ and σ in $\ell_1(S)$, we have

$$\begin{aligned}[(f*\phi)*\sigma](\zeta) &= [f*\phi](\sigma\zeta) \\ &= f[\phi(\sigma\zeta)] \\ &= f[(\phi\sigma)\zeta] \\ &= [f*\phi\sigma](\zeta),\end{aligned}$$

for every ζ in $\ell_1(S)$. Thus $(f*\phi)*\sigma = f*(\phi\sigma)$. It follows that, for each μ in $\ell_1(S)^{**}$, f in $\ell_1(S)^*$ and ϕ in $\ell_1(S)$,

$$\begin{aligned}[(\mu*f)*\phi](\sigma) &= \mu*f(\phi\sigma) \\ &= \mu(f*\phi\sigma) \\ &= \mu[(f*\phi)*\sigma] \\ &= [\mu*(f*\phi)](\sigma),\end{aligned}$$

for every σ in $\ell_1(S)$. Hence $(\mu*f)*\phi = \mu*(f*\phi)$. Then, this equality implies that

$$\begin{aligned}[(\lambda*\mu)*f](\phi) &= \lambda*\mu(f*\phi) \\ &= \lambda[\mu*(f*\phi)] \\ &= \lambda[(\mu*f)*\phi] \\ &= \lambda[(\mu*f)*\phi] \\ &= [\lambda*(\mu*f)](\phi),\end{aligned}$$

for all λ and μ in $\ell_1(S)^{**}$ and each f in $\ell_1(S)^*$ and for every ϕ in $\ell_1(S)$. Hence, (2.4.1) holds.

2.5. <u>Theorem</u>. *The Banach space* $\ell_1(S)^{**}$ *together with the Arens product forms a Banach algebra. Moreover, the natural embedding* $Q:\ell_1(S) \to \ell_1(S)^{**}$ *is an isometric isomorphism from the Banach* *algebra* $\ell_1(S)$ *into* $\ell_1(S)^{**}$. *That is,*

(2.5.1) Q *is one-to-one and linear;*

(2.5.2) $\|Q\phi\| = \|\phi\|$;

(2.5.3) $Q(\phi\sigma) = Q\phi * Q\sigma$,

for all ϕ *and* σ *in* $\ell_1(S)$.

Proof. Let ω,λ and μ be elements in $\ell_1(S)^{**}$. By (2.4.1) of Lemma 2.4, we have

$$\begin{aligned}[(\omega*\lambda)*\mu](f) &= [\omega*\lambda](\mu*f)\\ &= \omega[\lambda*(\mu*f)]\\ &= \omega[(\lambda*\mu)*f]\\ &= [\omega*(\lambda*\mu)](f),\end{aligned}$$

for each f in $\ell_1(S)^*$. Hence, the Arens product is associative.

For the distributivity of the Arens product, we first claim that, for all λ and μ in $\ell_1(S)^{**}$ and each f in $\ell_1(S)^*$,

$$(\lambda+\mu)*f = \lambda*f+\mu*f.$$

If ϕ is in $\ell_1(S)$, then

$$\begin{aligned}[(\lambda+\mu)*f](\phi) &= [\lambda+\mu](f*\phi)\\ &= \lambda(f*\phi)+\mu(f*\phi)\\ &= [\lambda*f+\mu*f](\phi).\end{aligned}$$

Hence, the assertion is true. Now, for all ω,λ and μ in $\ell_1(S)^{**}$, from the above claim, we have

$$[\omega*(\lambda+\mu)](f) = \omega[(\lambda+\mu)*f]$$
$$= \omega[\lambda*f+\mu*f]$$
$$= \omega(\lambda*f)+\omega(\mu*f)$$
$$= [\omega*\lambda+\omega*\mu](f)$$

and

$$[(\omega+\lambda)*\mu](f) = [\omega+\lambda](\mu*f)$$
$$= \omega(\mu+f)+\lambda(\mu*f)$$
$$= [\omega*\mu+\lambda*\mu](f),$$

for all f in $\ell_1(S)^*$. Thus the distributivity of the Arens product is proved.

For the property (2.1.3) of Definition 2.1, we let λ and μ be in $\ell_1(S)^{**}$ and α be a real number. From (2.4.7), it follows that

$$[\alpha(\lambda*\mu)](f) = [\lambda*\mu](\alpha f)$$
$$= \lambda[\mu*(\alpha f)]$$
$$= \lambda[\alpha(\mu*f)]$$
$$= [(\alpha\lambda)*\mu](f)$$

and

$$[\alpha(\lambda*\mu)](f) = \lambda[\mu*(\alpha f)]$$
$$= \lambda[(\alpha\mu)*f]$$
$$= [\lambda*(\alpha\mu)](f),$$

for every f in $\ell_1(S)^*$. Thus,

$$\alpha(\lambda*\mu) = (\alpha\lambda)*\mu = \lambda*(\alpha\mu).$$

Combining with (2.4.8) of Lemma 2.4, we have that $\ell_1(S)^{**}$ is a Banach algebra.

Finally, since Q is an isometric isomorphism from the Banach space $\ell_1(S)$ into $\ell_1(S)^{**}$, it only needs to prove condition (2.5.3). For all ϕ and σ in $\ell_1(S)$ and f in $\ell_1(S)^*$,

$$\begin{aligned} Q(\phi\sigma)(f) = f(\phi\sigma) &= [f*\phi](\sigma) \\ &= Q\sigma(f*\phi) \\ &= Q\sigma*f(\phi) \\ &= Q\phi(Q\sigma*f) \\ &= Q\phi*Q\sigma(f). \end{aligned}$$

This completes the proof.

2.6. Remark. In view of the last theorem and Theorem I.2.2, we have that $m(S)^*$ is a Banach algebra with the Arens product as multiplication. Furthermore, if f is in $m(S)$ and s in S, then

$$f*1_s(t) = f(st) = \ell_s f(t); \quad \text{and}$$

$$Q1_s*f(t) = Q1_s(\ell_t f) = f(ts) = \gamma_s f(t),$$

for all t in S. Hence, we have

(2.6.1) $\qquad f*1_s = \ell_s f \quad \text{and} \quad Q1_s*f = \gamma_s f$

It follows that, for each μ in $m(S)^*$ and f in $m(S)$, (2.6.2) $\ell_s^*\mu(f) = \mu*f(s)$ and $\gamma_s^*\mu(f) = \mu*Q1_s(f)$,

for each s in S.

2.7. Definition. Let ϕ be in $\ell_1(S)$. We define two linear operators ℓ_ϕ and γ_ϕ from $m(S)$ into itself by,

(2.7.1) $\ell_\phi(f)(s) = Q\phi(\gamma_s f)$; and

(2.7.2) $\gamma_\phi(f)(s) = Q\phi(\ell_s f)$,

for each s in S and each f in $m(S)$.

2.8. Remark. We remark that the above definitions are well-defined. Observe that $|\ell_\phi(f)(s)| = |Q\phi(\gamma_s f)| \le \|f\|\,\|\phi\|$ and $|\gamma_\phi(f)(s)| = |Q\phi(\ell_s f)| \le \|f\|\,\|\phi\|$. Hence, $\gamma_\phi f$ and $\ell_\phi f$ are in $m(S)$. The linearity of ℓ_ϕ and γ_ϕ follows directly from the linearity of $Q\phi$, ℓ_s and γ_s. Hence, ℓ_ϕ and γ_ϕ are linear operators. Furthermore,

(2.8.1) $\|\ell_\phi\| \le \|\phi\|$; and

(2.8.2) $\|\gamma_\phi\| \le \|\phi\|$,

for each ϕ in $\ell_1(S)$.

2.9. Lemma. *Let* f *be in* $m(S)$ *and* ϕ *in* $\ell_1(S)$. *Then we have*

(2.9.1) $$\ell_\phi(f) = f*\phi$$

Proof. If ϕ is in $\ell_1(S)$ and f is in $m(S)$, then by (2.6.1) we have

$$\ell\phi f(s) = Q\phi(\gamma_s f)$$

$$= Q\phi(Q1_s * f)$$

$$= Q\phi 1_s(f)$$

$$= f*\phi(s).$$

This completes the proof.

2.10. <u>Theorem</u>. *Let* ϕ *in* $\ell_1(S)$ *and* λ *in* $m(S)^*$ *be fixed. Then the mappings,*

(2.10.1) $\mu \to Q\phi*\mu$; *and*

(2.10.2) $\mu \to \mu*\lambda$,

are ω^*-ω^**-continuous from* $m(S)^*$ *into itself.*

<u>Proof</u>. To prove that $\mu \to Q\phi*\mu$ is ω^*-ω^*-continuous, we first claim that $\ell_\phi^*\mu = Q\phi*\mu$, for each μ in $m(S)^*$. Let f in $m(S)$ be arbitrary. Then the assertion follows from the following equalities:

$$\ell_\phi{}^*\mu(f) = \mu[\ell_\phi(f)]$$

$$= \mu[f*\phi]$$

$$= \mu*f(\phi)$$

$$= Q\phi(\mu*f)$$

$$= Q\phi*\mu(f).$$

Hence, by Proposition I.1.9, the adjoint operator $\mu \to Q\phi*\mu = \ell_\phi^*\mu$ is ω^*-ω^*-continuous.

Finally, to prove that $\mu \to \mu*\lambda$ is ω^*-ω^*-continuous, let $\{\mu_n\}$ be a net in $m(S)^*$ such that $\omega^*\text{-}\lim_n \mu_n = \mu$. Then, for each f in $m(S)$,

$$\lim_n \mu_n*\lambda(f) = \lim_n \mu_n(\lambda*f)$$
$$= \mu(\lambda*f)$$
$$= \mu*\lambda(f)$$

and this proves that $\mu \to \mu*\lambda$ is ω^*-ω^*-continuous.

2.11. <u>Lemma</u>. *If* λ *in* $m(S)^*$ *satisfies that* $\ell_s^*\lambda = \lambda$, *for each* s *in* S, *then*

$$\mu*\lambda = \mu(1)\lambda,$$

for all μ *in* $m(S)^*$.

<u>Proof</u>. Suppose that λ satisfies that $\ell_s^*\lambda = \lambda$, for each s in S. Observe that, for each f in $m(S)$, $\lambda*f(s) = \lambda(\ell_s f) = \lambda(f)$. Hence $\lambda*f = \lambda(f)1$. It follows that, for each ϕ in $\ell_1(S)$ and f in $m(S)$,

$$\mu*\lambda(f) = \mu(\lambda(f)1)$$
$$= \mu(1)\lambda(f).$$

This finishes the proof.

2.12. <u>Theorem</u>. *Let* M *be the set of all means on* m(S) *and let* LIM *and* RIM *be the set of all left and right invariant means, respectively. Then, with respect to the Arens product,*

(2.12.1) M *is a semigroup;*

(2.12.2) *If* LIM *is non-void, then* LIM *forms a two-sided ideal of* M, *i.e.,* $LIM*M \subseteq LIM$ *and* $M*LIM \subseteq LIM$. *Moreover, for each* λ *in* LIM *and* μ *in* M,

$$\mu*\lambda = \lambda;$$

(2.12.3) *If* RIM *is non-void, then* RIM *forms a left ideal of the semigroup* M, *that is*

$$M*RIM \subseteq RIM;$$

(2.12.4) *If* LIM *and* RIM *are non-void; then* LIM*RIM *consists of two-sided invariant means.*

<u>Proof</u>. To prove that M forms a semigroup, it is sufficient to show that M is closed under the Arens product. Let μ and λ be in M. Since $1*1_s = 1$, for all s in S, we have $\mu*1(s) = \mu(1*1_s) = \mu(1) = 1$. This implies that $\lambda*\mu(1) = \lambda(\mu*1) = \lambda(1) = 1$. Furthermore,

$$1 = |\lambda*\mu(1)| \leq \|\lambda*\mu\| \leq \|\lambda\| \, \|\mu\| = 1.$$

Hence, $\lambda*\mu(1) = \|\lambda*\mu\| = 1$, and, by Proposition 1.2, $\lambda*\mu$ is a mean. Consequently, M is closed under the Arens product.

Suppose that LIM is non-void. Let λ be a left invariant mean and μ in M. Then, by Lemma 2.11, we have $\mu*\lambda = \mu(1)\lambda = \lambda$ and hence $M*LIM \subseteq LIM$. Furthermore, to prove that $\lambda*\mu$ is also in LIM, we first claim that $(\mu*f)*1_s = \mu*(f*1_s)$, for each f in m(S) and each s in S. Let t be in S. Then by (2.6.2) of Remark 2.6, we have

$$[\mu*(f*1_s)](t) = \mu[f*1_{st}] = \mu*f(st) = [(\mu*f)*1_s](t),$$

and this proves our assertion. Therefore, for each s in S and each f in m(S),

$$\begin{aligned} \ell_s^*[\lambda*\mu](f) &= [\lambda*\mu](f*1_s) \\ &= \lambda[\mu*(f*1_s)] \\ &= \lambda[(\mu*f)*1_s] \\ &= \ell_s^*\lambda[\mu*f] \\ &= \lambda[\mu*f] \\ &= \lambda*\mu(f) \end{aligned}$$

and hence $\lambda*\mu$ is left invariant. Thus, we have $LIM*M \subseteq LIM$ and (2.12.2) is established.

Now, let μ be in RIM and λ in M. For each f in m(S) and all s and t in S, we have

$$\mu*\gamma_s f(t) = \mu[\ell_t(\gamma_s f)]$$

$$= \mu[\gamma_s(\ell_t f)]$$
$$= \mu[\ell_t f]$$
$$= \mu * f(t)$$

and hence $\mu * \gamma_s f = \mu * f$. It follows from the following equalities:

$$\gamma_s^*[\lambda*\mu](f) = [\lambda*\mu](\gamma_s f)$$
$$= \lambda[\mu*\gamma_s f]$$
$$= \lambda[\mu*f]$$
$$= \lambda*\mu(f)$$

that $\lambda*\mu$ is in RIM. Thus, $M*RIM \subseteq RIM$.

Finally, if RIM and LIM are non-void, then, by (2.12.2) and (2.12.3) that

$$LIM*RIM \subseteq LIM \cap RIM.$$

Hence, $LIM*RIM$ consist of two sided invariant means.

2.13. Corollary. *If a semigroup is both left and right amenable, then it is amenable.*

Proof. This is an immediate consequence of (2.12.4) of Theorem 2.12.

§3. Characterization of Amenable Semigroups.

Work has been done on characterizing amenable semigroups. In this section, we first collect all the necessary and sufficient conditions on a semigroup to be left amenable. Similar results for the case of right amenable semigroups holds with some minor exceptions. Finally, we prove the "Følner's condition" and "strong Følner's condition" for amenable semigroups.

Before we go into the main theme of this section, we give some necessary definitions.

3.1. Definition. A representation [or anti-representation] T of a semigroup S over a normed space X is a semigroup homomorphism [or anti-homomorphism] $s \to T_s$ from S into the semigroup $L(X,X)$, of all bounded linear operators from X into X. That is,

$$T_{st} = T_s T_t \quad [\text{or} \quad T_{st} = T_t T_s],$$

for all s and t in S. If $\sup\{\|T_s\| : s \text{ in } S\} < \infty$, then T is called bounded.

Furthermore, if $X = m(S)$ and T_s is ℓ_s or γ_s, for s in S, then they are called regular left or right representation respectively.

Let T be a representation (or anti-representation) of S over a normed space X. For each element ϕ in $\ell_1(S)$ with finite support, we define a linear operator T_ϕ on X by

$$T_\phi = \sum_{s \varepsilon S} \phi(s) T_s.$$

In particular, for the regular left and right representation, γ_ϕ and ℓ_ϕ coincide with those in Definition 2.7.

3.2. Lemma. *Let* T *be a bounded representation [or anti-representation] of* S *over a normed space* X. *Then, for each* ϕ *and* σ *in* $\ell_1(S)$ *with finite supports and for each scalar* α,

(3.2.1) $T_{\phi+\sigma} = T_\phi + T_\sigma$;

(3.2.2) $T_{\alpha\phi} = \alpha T_\phi$;

(3.2.3) $\| T_\phi \| \le M \| \phi \|$,

where $M = \sup\{\| T_s \| : s \text{ in } S\}$;

(3.2.4) $T_{\phi\sigma} = T_\phi T_\sigma$ [or $T_{\phi\sigma} = T_\sigma T_\phi$].

Proof. The conditions (3.2.1) and (3.2.2) follows directly from the definition.

Since $M = \sup\{\| T_s \| : s \text{ in } S\}$, we have

$$\begin{aligned} \| T_\phi \| &= \| \textstyle\sum_{s\varepsilon S} \phi(s) T_s \| \\ &\le \textstyle\sum_{s\varepsilon S} |\phi(s)| \, \| T_s \| \\ &\le \| \phi \| M, \end{aligned}$$

and hence (3.2.3) holds.

Finally, for (3.2.4), we prove only the case of anti-representation. Since the support of $\phi\sigma$ is contained in the product of the supports of ϕ and of σ, $\phi\sigma$ has finite support and $T_{\phi\sigma}$ is defined. Hence

$$\begin{aligned} T_{\phi\sigma} &= \textstyle\sum_{x\varepsilon S} \phi\sigma(x) T_x \\ &= \textstyle\sum_{x\varepsilon S} \sum_{st=x} \phi(s)\sigma(t) T_{st} \end{aligned}$$

$$= \sum_{st\varepsilon S}\phi(s)\sigma(t)T_tT_s$$

$$= (\sum_{t\varepsilon S}\sigma(t)T_t)(\sum_{s\varepsilon S}\phi(s)T_s)$$

$$= T_\sigma T_\phi$$

and this establishes (3.2.4).

Let (E,τ) be a locally convex space. For each subset A of E, we denote by $C_0(A)$ the convex hull of A; by $\tau\text{-}C\ell(A)$ the τ-closure of A. We will write $\tau\text{-}C\ell C_0(A)$ for $\tau\text{-}C\ell(C_0(A))$. If E is a normed space, we use $C\ell(A)$ for the uniform closure of A.

3.3. Notation. Let T be a representation [or anti-representation] of S over a normed space X.

(3.3.1) For each x in X, by the orbit of x in X, we mean the set

$$O(x) = \{T_s(x) : s \text{ in } S\}.$$

In particular, for the regular left and right representation, we write, for each f in $m(S)$, write $LO(f)$ for $\{\ell_s f : s \text{ is in } S\}$ and $RO(f)$ for $\{\gamma_s f : s \in S\}$, respectively.

(3.3.2) Let K_X denote the linear span of the set $\{x-T_s(x) : x \text{ is in } X \text{ and } s \text{ is in } S\}$. For the regular left representation, we write K for $K_{m(S)}$.

3.4. Definition. A semigroup S is called right stationary if, for each f in m(S), there is at least one constant function in $\omega^*\text{-}C\ell C_0 RO(f)$.

3.5. Lemma. *Let* S *be a right stationary semigroup. Then, for each* a *in* S *and* f *in* m(S), *there is a mean* μ *on* m(S) *such that, for each* s *in* S,

(3.5.1) $\mu[\ell_s(f-\ell_a f)] = 0;$

(3.5.2) $\mu*(f-\ell_a f) = 0.$

Proof. Let f in m(S) and a in S be arbitrary. Since S is right stationary, there is a net $\{\phi_n\}$ of finite means such that $\omega^*\text{-}\lim_n \gamma_{\phi_n} f = c1$, for some constant c. For each s in S,

$$\lim_n \gamma_{\phi_n}(f-\ell_a f)(s) = \lim_n [\gamma_{\phi_n} f(s) - \gamma_{\phi_n} f(as)]$$
$$= c - c$$
$$= 0.$$

By Proposition 1.7 and Theorem 1.6, there is subnet $\{\phi_k\}$ of $\{\phi_n\}$ and a mean μ such that $\omega^*\text{-}\lim_k Q\phi_k = \mu$. Then, for each s in S,

$$\mu[\ell_s(f-\ell_a f)] = \lim_k Q\phi_k[\ell_s(f-\ell_a f)]$$
$$= \lim_k \gamma_{\phi_k}(f-\ell_a f)(s)$$
$$= 0$$

and this proves (3.5.1).

To establish (3.5.2), we observe that, for each s in S,

$$\begin{aligned} \mu*(f-\ell_a f)(s) &= \mu[(f-\ell_a f)*1_s] \\ &= \mu[\ell_s(f-\ell_a f)] \\ &= 0. \end{aligned}$$

This completes the proof.

By an affine map F on a vector space E into E itself, we mean a function F on E that satisfies

$$F[\alpha x+(1-\alpha)y] = \alpha F(x)+(1-\alpha)F(y),$$

whenever x and y are in E and $0 \leq \alpha \leq 1$.

3.6. <u>Theorem</u>. *Let* S *be a semigroup. Then the following conditions are equivalent:*

(3.6.1) S *is left amenable;*

(3.6.2) *there is a net* $\{\phi_n\}$ *of finite means on* $m(S)$ *that is* ω*-convergent to left invariance;*

(3.6.3) *there is a net* $\{\phi_n\}$ *of finite means on* $m(S)$ *that is norm-convergent to left invariance;*

(3.6.4) *for every anti-representation* $\{T_s : s \text{ in } S\}$ *of* S *over a normed space* X *with* $\|T_s\| \leq 1$ *for each* $s \in S$,

$$\text{dist}(0,\overline{C_0 O(x)}) = \text{dist}(0,x+K_X),$$

for every x *in* X;

(3.6.5) $\operatorname{dist}(0,1+K) = 1$;

(3.6.6) $\inf\{h(s): s \text{ is in } S\} \leq 0$,

for every h *in* K;

(3.6.7) $\sup\{h(s): s \text{ is in } S\} \geq 0$

for every h *in* K;

(3.6.8) *there is a net* $\{\phi_n\}$ *of finite means on* S *such that, for all* f *in* $m(S)$, $\{\gamma_{\phi_n} f\}$ *converges pointwise to a constant function;*

(3.6.9) S *is right stationery;*

(3.6.10) *every representation of* S *as continuous affine maps from a compact convex set in a locally convex space into itself has a common fixed point.*

Proof. We prove the theorem by proving (3.6.1) → (3.6.2) → (3.6.3) → (3.6.4) → (3.6.5) → (3.6.1);(3.6.1) → (3.6.6) → (3.6.7) → (3.6.5); (3.6.1) → (3.6.8) → (3.6.9) → (3.6.1); and (3.6.8) → (3.6.10) → (3.6.1).

That (3.6.1) implies (3.6.2) follows immediately from Propositions 1.7 and 1.11.

To show (3.6.2) implies (3.6.3), let $E_s = \ell_1(S)$, for each s in S. Then, by Proposition I.1.8, $E = \prod_{s \in S} E_s$ with the product topology τ of the norm-topologies on E_s is a locally convex space and the ω-topology $\sigma(E,E^*)$ of E is exactly the product topology of the ω-topology $\sigma(E_s,E_s^*)$ of E_s. Define a linear transformation $T:\ell_1(S) \to E$ by

$$T(\phi) = (\phi - l_s\phi)_{s\varepsilon S},$$

for each ϕ in $\ell_1(S)$. The linearity of T follows from the fact that $\ell_1(S)$ is a Banach algebra. Since the set Φ of all finite means is convex, $T[\Phi]$ is convex. By Proposition I.1.7, we have $\tau\text{-}C\ell(T[\Phi]) = \omega\text{-}C\ell(T[\Phi])$. The condition (3.6.2) implies that 0 is in $\omega\text{-}C\ell(T[\Phi])$ and hence in $\tau\text{-}C\ell(T[\Phi])$. Thus, there is a net $\{\phi_n\}$ in Φ such that $\{T(\phi_n)\}$ converges to 0 in τ. It follows from the continuity of the projections that $\{\phi_n - l_s\phi_n\}$ converges to 0 in the norm-topology of $\ell_1(S)$, for each s in S. This establishes (3.6.3). This result is originally due to M. M. Day [5]. But his proof is complicated and the foregoing proof is due to Namioka [26].

Now, to prove (3.6.3) implies (3.6.4), recall that, for all subset A and B in a normed space X, $\text{dist}(A,B) = \inf\{\|a-b\|: a \text{ in } A \text{ and } b \text{ in } B\}$. Let x in X be arbitrary. For each $y = \sum_{s\varepsilon S}\phi(s)T_s(x)$ in $C_0O(x)$, it follows from the following sets of equalities

$$\begin{aligned} y &= x - x + y \\ &= x - \textstyle\sum_{s\varepsilon S}\phi(s)x + \sum_{s\varepsilon S}\phi(s)T_s(x) \\ &= x + \textstyle\sum_{s\varepsilon S}\phi(s)(T_s(x)-x) \end{aligned}$$

that $C_0O(x) \subseteq x + K_X$ and hence

(a) $$\operatorname{dist}(0,C_0O(x) \geq \operatorname{dist}(0,x+K_X).$$

To show the reverse inequality, let $\{\phi_n\}$ be a net of finite means such that $\lim_n \|\phi_n - 1_s\phi_n\| = 0$, for each s in S. For every $\varepsilon > 0$, let $y = \sum_{i=1}^m x_i - T_{s_i}(x_i)$, x_i in X and s_i in S, $1 \leq i \leq m$, be in K_X such that

$$\|x+y\| < \operatorname{dist}(0,x+K_X) + \varepsilon/2.$$

Let n be such that, for all $i = 1, 2, \ldots, m$,

$$\|\phi_n - 1_{s_i}\phi_n\| < \varepsilon/2mM,$$

where $M = \max\{\|x_i\|:1 \leq i \leq m\}$. By Lemma 3.2, we have

$$\begin{aligned}
\|T_{\phi_n}(y)\| &= \|T_{\phi_n}[\textstyle\sum_{i=1}^m x_i - T_{s_i}(x_i)]\| \\
&= \|\textstyle\sum_{i=1}^m T_{\phi_n}(x_i) - T_{\phi_n}T_{s_i}(x_i)]\| \\
&\leq \textstyle\sum_{i=1}^m \|T_{\phi_n - 1_{s_i}\phi_n}(x_i)\| \\
&\leq \textstyle\sum_{i=1}^m \|\phi_n - 1_{s_i}\phi_n\| \, \|x_i\| \\
&\leq \textstyle\sum_{i=1}^m \|\phi_n - 1_s\phi_n\| M \\
&< \varepsilon/2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|T_{\phi_n}(x)\| &= \|T_{\phi_n}(x+y) - T_{\phi_n}(y)\| \\
&\leq \|T_{\phi_n}(x+y)\| + \|T_{\phi_n}(y)\| \\
&< \|\phi_n\| \, \|x+y\| + \varepsilon/2
\end{aligned}$$

$$< \operatorname{dist}(0,x+K_X) + \varepsilon.$$

Thus, we have shown that, for every $\varepsilon > 0$, there is a $x_0 = T_{\phi_n}(x)$ in $C_0O(x)$ such that $\|x_0\| < \operatorname{dist}(0,x+K_X) + \varepsilon$. Consequently, we have

$$\text{(b)} \qquad \operatorname{dist}(0,C_0O(x)) = \inf\{\|y\| : y \text{ is in } C_0O(x)\} \leq \operatorname{dist}(0,x+K_X)$$

and, combining inequalities (a) and (b), (3.6.4) follows. This result is due to Glicksberg [12] and the proof above is due to Granirer [13].

To prove (3.6.4) implies (3.6.5), since $\{\ell_s : s \text{ is in } S\}$ is an anti-representation of S and $\|\ell_s\| \leq 1$ for each s, we have $1 = \operatorname{dist}(0,C_0O(1)) = \operatorname{dist}(0,1+K)$.

Now, we prove (3.6.5) implies (3.6.1). The condition (3.6.5) implies that 1 is not in K. By an application of the Hahn-Banach Theorem (See [7, Lemma 12, p. 64]), there is a μ in $m(S)^*$ such that $\mu(1) = 1$ and $\mu(h) = 0$ for all h in K, with $\|\mu\| = 1/\operatorname{dist}(0,1+K) = 1$. By Proposition 1.2, μ is a mean. Since $f - \ell_s f$ is in K, for each f in $m(S)$ and s in S, $\mu(f-\ell_s f) = 0$ and hence μ is a left invariant mean.

To prove that (3.6.1) implies (3.6.6), let μ be a left invariant mean. Then $\mu(h) = 0$, for all h in K. By the definition of means on $m(S)$, we have

$$\inf\{h(s) : s \text{ in } S\} \leq \mu(h) = 0,$$

for each h in K.

To see that (3.6.6) implies (3.6.7), we observe that $\sup\{f(s):s \text{ in } S\} = -\inf\{-f(s):s \text{ in } S\}$, for all f in m(S). Then, for each h in K, -h is in K and hence

$$\sup\{h(s):s \text{ in } S\} = -\inf\{-h(s):s \text{ in } S\} \geq 0.$$

This result is due to Dixmier [6].

Now, we show that (3.6.7) implies (3.6.5). Since $1 = 1+0$ is in $1+K$, $\text{dist}(0,1+K)$ is at least one. By (3.6.7), we have, for each h in K,

$$\begin{aligned} 1 &\leq 1 + \{\sup h(s):s \text{ in } S\} \\ &\leq \sup\{1+h(s):s \text{ in } S\} \\ &\leq \sup\{|1+h(s)|:s \text{ in } S\} \\ &= \|1+h\| \end{aligned}$$

and hence $1 \leq \text{dist}(0,1+K)$. This establishes (3.6.5).

To show that (3.6.1) implies (3.6.8), let μ be a left invariant mean. By Proposition 1.7, there is a net $\{\phi_n\}$ of finite means such that $\omega^*\text{-}\lim_n Q\phi_n = \mu$. Hence, for each f in m(S) and t in S,

$$\lim_n \gamma_{\phi_n} f(t) = \lim_n Q\phi_n(\ell_t f) = \mu(\ell_t f) = \mu(f).$$

Hence, this establishes (3.6.8).

That (3.6.8) implies (3.6.9) follows from Proposition I.2.4 that $\{\gamma_{\phi_n} f\}$ converges pointwise to constant if and only if it converges to a constant function in the ω^*-topology of $m(S)$. Then, (3.6.8) becomes formally stronger than (3.6.9).

Now, we prove (3.6.9) implies (3.6.1). For each f in $m(S)$ and each a in S, let

$$K(f,a) = \{\phi \text{ in } M: \phi[\ell_s(f-\ell_a f)] = 0, \text{ for all } s \text{ in } S\}.$$

By Lemma 3.5, $K(f,a)$ is non-void, for each f in $m(S)$ and each a in S. If $Y = \{f_1,\ldots,f_n\}$ is in $m(S)$ and $F = \{a_1,\ldots,a_n\}$ is in S, then define $K(Y,F) = \bigcap_{i=1}^{n} K(f_i,a_i)$. First we claim that $K(Y,F)$ is non-void. By Lemma 3.5, we know that if $n = 1$, then $K(Y,F)$ is not empty. Assume that there is a μ in $\bigcap_{i=1}^{n-1} K(f_i,a_i)$. Then, let λ be in $K(\mu*f_n,a_n)$. Observe that, for each ω in M, each g in $m(S)$ and each s in S, $\omega*\ell_s g = \ell_s(\omega*g)$. Hence, by (3.5.2) of Lemma 3.5, we have

$$\begin{aligned}\lambda*\mu[\ell_s(f_i-\ell_{a_i} f_i)] &= \lambda[\mu*\ell_s(f_i-\ell_{a_i} f_i)] \\ &= \lambda[\ell_s(\mu*f_i-\ell_{a_i} f_i)] \\ &= 0,\end{aligned}$$

for $1 \leq i \leq n-1$, and

$$\begin{aligned}\lambda*\mu[\ell_s(f_n-\ell_{a_n} f_n)] &= \lambda[\ell_s(\mu*f_n-\ell_{a_n}(\mu*f_n))] \\ &= 0,\end{aligned}$$

for every s in S. Thus, $\lambda*\mu$ is in $K(Y,F)$. Consequently, the family $\mathcal{K} = \{K(f,a): f \text{ in } m(S),\ a \text{ in } S\}$ has finite intersection property. Let f be in $m(S)$ and a in S. For each s in S, define $h_s = \ell_s(f-\ell_a f)$. Then h_s is a ω^*-continuous linear functional on $m(S)^*$. It follows that $K(f,a) = M \cap (\bigcap_{s\in S} h_s^{-1}(0))$ is ω^*-closed in M. By the ω^*-compactness of the set M, $\mathcal{C}$ has non-empty intersection. Let μ be in $\bigcap\{c: c \text{ is in } \mathcal{C}\}$. For each f in m(S) and s in S, μ is in K(f,s) and by (3.5.2) of Lemma 3.5,

$$\mu*\mu(f-\ell_s f) = \mu[\mu*(f-\ell_s f)] = 0$$

Hence, $\mu*\mu$ is a left invariant mean. Moreover, for any λ in M, $\lambda*\mu$ will be a left invariant mean also. This result is due to Mitchell [25] and the above proof is due to Granirer and Lau [14].

We now prove that (3.6.8) implies (3.6.10). Suppose that $\{\phi_n\}$ is a net of finite means such that $\{\gamma_{\phi_n} f\}$, for each f in m(S), converges pointwise to a constant. Let S be represented as continuous affine maps from a compact convex set C in a locally convex space E into C itself. For each ϕ_n, we define the affine map F_n by $F_n(x) = \sum_{s\in S}\phi_n(s)s(x)$, for each x in C. Since C is convex, $F_n(x)$ is in C, for each x in C. Hence F_n is well-defined. Fixed a y in C. By the compactness of C, there is a subnet $\{F_k(y)\}$ of $\{F_n(y)\}$ and a y_0 in C such that $\lim_k F_k(y) = y_0$. (Note that, for each f in m(S), the corresponding subnet $\{\gamma_{\phi_k} f\}$ converges pointwise to a constant function also.) For each λ in E*, we define

f_λ on S by $f_\lambda(s) = \lambda(s(y))$, for each s in S. Since C is compact and λ is continuous, f_λ is in $m(S)$. Also, for each t in S and each k,

$$\begin{aligned}\gamma_{\phi_k} f\ (t) &= \textstyle\sum_{s\varepsilon S}\phi_k(s)\gamma_s f_\lambda(t)\\ &= \textstyle\sum_{s\varepsilon S}\phi_k(s) f_\lambda(ts)\\ &= \textstyle\sum_{s\varepsilon S}\phi_k(s)\lambda(ts(y))\\ &= \lambda[\textstyle\sum_{s\varepsilon S}\phi_k(s)ts(y)]\\ &= \lambda[t(\textstyle\sum_{s\varepsilon S}\phi_k(s)s(y)]\\ &= \lambda[t(F_k(y))].\end{aligned}$$

Since $\{\gamma_{\phi_k} f_\lambda\}$ converges pointwise to a constant, say α, we have

$$\begin{aligned}\alpha &= \lim_k \ \gamma_{\phi_k} f_\lambda(t)\\ &= \lim_k \lambda[t(F_k(y))]\\ &= \lambda[t(y_0)].\end{aligned}$$

Hence, for any s_0 in S, $[s_0(y_0)] = [s(s_0(y_0))] = \alpha$, for each s in S. Since E is locally convex, it has enough continuous linear functionals to seperate points. Therefore, it follows that $s(s_0(y_0)) = s_0(y_0)$, for each s in S, and hence $s_0(y_0)$ is a common fixed point for S. This result is due to Day [5] and the foregoing proof is due to Mitchell [25].

To see (3.6.10) implies (3.6.1), observe that $\{\ell_s^*: s$ is in $S\}$ is a representation of S as continuous affine maps from the ω^*-compact convex set $\bar{M}$ in $m(S)^*$ into $\bar{M}$ itself. Then the common fixed point is a left invariant mean on m(S).

For any finite set A, we denote by $|A|$ the number of elements in A.

3.7. Definition. A semigroup S is said to satisfy the Følner's condition if, for each finite subset F of S and each $\varepsilon > 0$, there is a finite subset A of S such that

(3.7.1) $$|sA \setminus A| < \varepsilon |A|,$$

for each s in S.

In [9], Følner showed that a group is left amenable if and only if it satisfies the Følner's condition. This condition was generalized to left amenable semigroups as a necessary condition by Frey in his thesis [10]. His proof was complicated and here we bring in Namioka's elegant proof.

Let S be a semigroup throughout this section. For each finite set A in S, define a finite mean μ_A on S by $\mu_A = \frac{1}{|A|} 1_A$, where 1_A is the characteristic function of A. Such finite means are called arithmatic means.

3.8. Lemma *Let* ϕ *be a finite mean on* S. *Then there is a* *finite family* $\{A_1, A_2, \ldots, A_n\}$ *of finite sets in* S *with* $A_{i+1} \subseteq A_i$, $1 \leq i \leq n$, *and a finite set* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *of positive real numbers with* $\sum_{i=1}^n \lambda_i = 1$ *such that*

$$\phi = \sum_{i=1}^n \lambda_i \mu_{A_i}.$$

<u>Proof</u>. Suppose ϕ is a finite mean on S. Let $0 < a_1 < a_2 < \ldots < a_n$ be the distinct values of ϕ. Let $A_i = \{s \text{ in } S : a_i \leq \phi(s)\}$, $1 \leq i \leq n$. Then each A_i is finite and $A_{i+1} \subseteq A_i$, $1 \leq i \leq n$. Since $|A|\mu_A = 1_A$,

$$\begin{aligned}\phi &= a_1 1_{A_1} + (a_2 - a_1) 1_{A_2} \ldots + (a_n - a_{n-1}) 1_{A_n} \\ &= a_1 |A_1| \mu_{A_1} + (a_2 - a_1) |A_2| \mu_{A_2} + \ldots + (a_n - a_{n-1}) |A_n| \mu_{A_n} \\ &= \sum_{i=1}^n \lambda_i \mu_{A_i},\end{aligned}$$

where $\lambda_1 = a_1 |A_1|$ and $\lambda_i = (a_i - a_{i-1})|A_i|$, $2 \leq i \leq n$. Since $0 < a_1 < \ldots < a_n$, we have $\lambda_i > 0$, $1 \leq i \leq n$. Furthermore,

$$\begin{aligned}1 &= \sum_{s \varepsilon S} \phi(s) \\ &= \sum_{s \varepsilon S} \sum_{i=1}^n \lambda_i \mu_{A_i}(s) \\ &= \sum_{i=1}^n \lambda_i (\sum_{s \varepsilon S} \mu_{A_i}(s) \\ &= \sum_{i=1}^n \lambda_i.\end{aligned}$$

This finishes the proof.

For s and t in S, we denote by $s^{-1}t$ the set consisting of all x in S such that $sx = t$.

3.9. Lemma. *Let* A *be a finite subset of* S. *Then, for each* s *in* S,

$$\begin{aligned}
[1_s\mu_A - \mu_A](t) &= |A \cap s^{-1}t|/|A| && \text{if } t \in sA \setminus A \\
&= -1/|A| && \text{if } t \in A \setminus sA \\
&= (|A \cap s^{-1}t| - 1)/|A| && \text{if } t \in A \cap sA \\
&= 0 && \text{otherwise}.
\end{aligned}$$

Proof. Observe that, for s and t in S,

$$\begin{aligned}
1_s\mu_A(t) &= \textstyle\sum_{sx=t}\mu_A(x) \\
&= \textstyle\sum_{x\in s^{-1}t}\mu_A(x) \\
&= |A \cap s^{-1}t|/|A|.
\end{aligned}$$

If t is in $sA \setminus A$, then $t = sa$, for some a in A and sa is not in A. Hence

$$[1_s\mu_A - \mu_A](t) = (|A \cap s^{-1}t|/|A|) - 0 = |A \cap s^{-1}t|/|A|.$$

Suppose that t is in $A \setminus sA$. Then $1_s\mu_A(t) = 0$ and $\mu_A(t) = 1/|A|$. Hence, $[1_s\mu_A - \mu_A](t) = -1/|A|$. Finally, if t is in $A \cap sA$, then

$$\begin{aligned}
[1_s\mu_A - \mu_A](t) &= |A \cap s^{-1}t|/|A| - 1/|A| \\
&= (|A \cap s^{-1}t| - 1)/|A|.
\end{aligned}$$

This completes the proof.

3.10. Corollary. *Let* A *be the same as in* Lemma 3.9. *Then, for each* s *in* S,

$$\| 1_s\mu_A - \mu_A \| = 2|A\setminus sA|/|A|.$$

Proof. From Lemma 3.9, we have

$$\begin{aligned}
\| 1_s\mu_A - \mu_A \| &= \textstyle\sum_{t\in S} |[1_s\mu_A - \mu_A](t)| \\
&= \textstyle\sum_{t\in sA\setminus A} |[1_s\mu_A - \mu_A](t)| \\
&+ \textstyle\sum_{t\in sA\cap A} |[1_s\mu_A - \mu_A](t)| \\
&+ \textstyle\sum_{t\in A\setminus sA} |[1_s\mu_A - \mu_A](t)| \\
&= \textstyle\sum_{t\in sA\setminus A} |A\cap s^{-1}t|/|A| \\
&+ \textstyle\sum_{t\in sA\cap A} (|A\cap s^{-1}t|-1)/|A| \\
&+ \textstyle\sum_{t\in A\setminus sA} \frac{1}{|A|} \\
&= \textstyle\sum_{t\in sA\cap A} |A\cap s^{-1}t|/|A| + \sum_{t\in sA\cap A} |A\cap s^{-1}A|/|A| \\
&- \textstyle\sum_{t\in sA\cap A} \frac{1}{|A|} + \sum_{t\in A\setminus sA} \frac{1}{|A|} \\
&= \textstyle\sum_{t\in sA} |A\cap s^{-1}t|/|A| - |sA\cap A|/|A| \\
&+ |A\setminus sA|/|A| \\
&= 1 - |A\cap sA|/|A| + |A\setminus sA|/|A| \\
&= (|A| - |A\cap sA| + |A\setminus sA|)/|A| \\
&= (|A\cap sA| + |A\setminus sA| + |A\cap sA| + |A\setminus sA|)/|A|
\end{aligned}$$

$$= 2|A\setminus sA|/|A|,$$

for each s in S. This proves the Corollary.

3.11. Lemma. *Let* ϕ *in* Φ *be expressed as in* Lemma 3.8. *Then, for each* s *in* S,

$$\| 1_s\phi-\phi\| \geq \sum_{i=1}^{n} \lambda_i |sA_i\setminus A_i|/|A_i|.$$

Proof. Let s in S be arbitrary. Since $\phi = \sum_{i=1}^{n}\lambda_i\mu_{A_i}$ as in Lemma 3.8, we have

$$1_s\phi-\phi = \sum_{i=1}^{n}\lambda_i(1_s\mu_{A_i}-\mu_{A_i})$$

Let $B = \bigcup_{j=1}^{n}(A_j\setminus sA_j)$. Then, by Lemma 3.9, for each i, $1 \leq i \leq n$, $\lambda_i(1_s\mu_{A_i}-\mu_{A_i})(t) \geq 0$, if t is in $S\setminus B$. Hence, $1_s\phi(t)-\phi(t) \geq 0$. For every i and j, either $A_i \subset A_j$ or $A_j \subset A_i$, and hence $sA_i \subset sA_j$ or $sA_j \subset sA_i$. Therefore

$$(sA_i\setminus A_i) \cap (A_j\setminus sA_j) = \phi,$$

for all i and j, $1 \leq i$, $j \leq n$. But for each i, $sA_i\setminus A_i \subseteq S\setminus B$. Consequently,

$$\begin{aligned}
\| 1_s\phi-\phi \| &= \sum_{t\in S}|1_s\phi(t)-\phi(t)| \\
&\geq \sum_{t\in S\setminus B}(1_s\phi(t)-\phi(t)) \\
&= \sum_{i=1}^{n}\lambda_i(\sum_{t\in S\setminus B}[1_s\mu_{A_i}-\mu_{A_i}](t)) \\
&\geq \sum_{i=1}^{n}\lambda_i(\sum_{t\in sA_i\setminus A_i}[1_s\mu_{A_i}-\mu_{A_i}](t))
\end{aligned}$$

By Lemma 3.9, it follows that

$$\begin{aligned}\|1_s\phi-\phi\| &\geq \textstyle\sum_{i=1}^n \lambda_i \sum_{t\in sA_i\setminus A_i}[1_s\mu_{A_i}-\mu_{A_i}](t)) \\ &\geq \textstyle\sum_{i=1}^n \lambda_i (\sum_{t\in sA_i\setminus A_i} |A_i\cap s^{-1}t|/|A_i|) \\ &\geq \textstyle\sum_{i=1}^n \lambda_i |sA_i\setminus A_i|/|A_i|.\end{aligned}$$

This finishes the proof.

3.12. <u>Theorem</u>. *If* S *is left amenable, then* S *satisfies the Følner's condition.*

<u>Proof</u>. Let $F = \{s_1, s_2, \ldots, s_k\}$. Since S is left amenable, by Theorem 3.6, there is a finite mean ϕ on S such that $\|1_s\phi-\phi\| < \varepsilon/k$, for all s in F. Let $\phi = \sum_{i=1}^n \lambda_i \mu_{A_i}$ as in Lemma 3.8. Then it follows from Lemma 3.11 that

$$\varepsilon/k > \|1_{s_j}\phi-\phi\| \geq \textstyle\sum_{i=1}^n \lambda_i |s_jA_i\setminus A_i|/|A_i|,$$

for $1 \leq j \leq k$. Choose $A = A_{i_0}$ such that

$$\textstyle\sum_{j=1}^k |s_jA_{i_0}\setminus A_{i_0}|/|A_{i_0}| = \min\limits_{1\leq i\leq n}\{\sum_{j=1}^k |s_jA_i\setminus A_i|/|A_i|\}.$$

Then, for each m, $1 \leq m \leq k$,

$$\begin{aligned}\varepsilon &> \textstyle\sum_{j=1}^k \|1_{s_j}\phi-\phi\| \\ &\geq \textstyle\sum_{j=1}^k \sum_{i=1}^n \lambda_i |s_jA_i\setminus A_i|/|A_i| \\ &= \textstyle\sum_{i=1}^n \lambda_i (\sum_{j=1}^k |s_jA_i\setminus A_i|/|A_i|) \\ &\geq \textstyle\sum_{i=1}^n \lambda_i (\sum_{i=1}^k |s_jA\setminus A|/|A|)\end{aligned}$$

$$= (\sum_{i=1}^{n}\lambda_i)(\sum_{j=1}^{k}|s_jA\backslash A|/|A|)$$

$$= \sum_{j=1}^{k}|s_jA\backslash A|/|A|$$

$$\geq |s_mA\backslash A|/|A|.$$

Thus, $|s_mA\backslash A| < \varepsilon|A|$, for all s_m in F.

For the case of right amenable semigroups, if we change the inequality $|sA\backslash A| < \varepsilon|A|$ in the Følner's condition to $|As\backslash A| < \varepsilon|A|$, then the last theorem still holds.

3.13. Remark. The Følner's condition is not sufficient for a semigroup to be left amenable. Since every finite semigroup satisfies the Følner's condition trivially (simply take $A = S$) but not every finite semigroup is left amenable. For example, take $S = \{s,t\}$ with $st = s = ss$ and $ts = t = tt$. Define f on S by $f(s) = 0$ and $f(t) = 1$. Then $h = [f-\ell_t f]-[f-\ell_s f] = \ell_s f-\ell_t f$ is in the linear span of $\{\ell_s f-f : s \in S$ and $f \in m(S)\}$. Since $\sup\{h(s): s$ in $S\} = -1 < 0$, by Theorem 3.6, S is not left amenable.

Now, we discuss a stronger condition on S which is sufficient for S to be left amenable.

3.14. Definition. A semigroup S is said to satisfy the strong Følner's condition if, for each finite subset F of S and each $\varepsilon > 0$, there is a finite subset A of S such that

$$|A \setminus sA| < \varepsilon |A|,$$

for all s in F.

3.15. Theorem. *If* S *satisfies the strong Følner's condition, then there is a net of arithmetic means that is norm-convergent to left invariance and hence* S *is left amenable.*

Proof. Let $\sum$ be the family consisting of pairs (n,F), where n is a natural number and F is a non-void finite subset of S. Define a partial order $\leq$ on $\sum$ by $(n,F) \leq (m,H)$ if and only if $F \subseteq H$ and $m \geq n$. For each $\sigma = (n,F)$ in $\sum$, let A_σ be such that $|A_\sigma \setminus sA_\sigma| < \frac{1}{n}|A_\sigma|$, for all s in F. Then, by Corollary 3.10, for each s in S and every $\varepsilon > 0$, there is a $\sigma_0 = (n_0,F_0)$ in $\sum$, where $\frac{2}{n_0} < \varepsilon$ and s in F_0, such that whenever $\sigma = (n,F) \geq \sigma_0$,

$$\begin{aligned} \| 1_s\mu_{A_\sigma} - \mu_{A_\sigma} \| &= 2|A_\sigma \setminus sA_\sigma| / |A_\sigma| \\ &< \frac{2}{n} \\ &\leq \frac{2}{n_0} \\ &< \varepsilon. \end{aligned}$$

Hence $\{\mu_{A_\sigma}\}\sigma\varepsilon\Sigma$ is norm-convergent to left invariance and this completes the proof.

3.16. Proposition. *In a left cancellative semigroup, the strong*

Følner's condition is equivalent to the Følner's condition.

Proof. Let A be a finite subset of a left cancellative semigroup S. For each s in S, we have $|sA| = |A|$. Hence

$$|sA \setminus A| = |sA| - |sA \cap A| = |A| - |sA \cap A| = |A \setminus sA|$$

and the proposition is now clear.

3.17. Corollary. *A left cancellative semigroup is left amenable if and only if it satisfies the strong Følner's condition.*

Proof. The corollary follows from Proposition 3.16 immediately.

3.18. Remark. The question whether or not every left amenable semigroup satisfies the strong Følner's condition depends on Sorenson's conjecture: Any right cancellative left amenable semigroup must also be left cancellative. (See [3, Theorem 6, p. 591.]) To see this, let S be a left amenable semigroup. Define a relation R on S by sRt, s and t in S, if and only if there is a u in S such that $su = tu$. Observe that any two right ideal of S intersect (see proposition 4.14 of next section) so that R is well-defined and is a congruence relation; i.e. R is an equivalent relation such that sRt implies $asRat$ and $saRta$, for each a in S. (See [22] and [12, Lemma 2, p. 371].) Then, the set S/R, of all congruence classes of R, with the binary operation defined by $\bar{x}\,\bar{y} = \overline{xy}$, for all $\bar{x}$ and

$\bar{y}$ in S/R, is a right cancellative left amenable semigroup. (See Proposition 4.3 of next section.) Assume that the Sorenson conjecture is valid. Then S/R is left cancellative and hence satisfies the strong Følner's condition. For each s in S, let $\bar{s}$ be the congruence class that contains s. Then, for each finite set F in S and each $\varepsilon > 0$, there is a finite set $\bar{A}$ in S/R such that $|\bar{A}\backslash\bar{s}\bar{A}| < \varepsilon|\bar{A}|$, for each s in F. Then, one can "lift" the set $\bar{A}$ back to A, say, in S such that $|A\backslash sA| < \varepsilon|A|$, for each s in F. Thus, S satisfies the strong Følner's condition. To see the "lifting" of the set $\bar{A}$, let A be a set of representatives of elements of $\bar{A}$. Observe that $|A\backslash sA| \geq |\bar{A}\backslash\bar{s}\bar{A}|$ and $|\bar{A}| = |A|$. If $|A\backslash sA| = |\bar{A}\backslash\bar{s}\bar{A}|$, for each s in F, then we are done. If there is an s in F such that $|A\backslash sA| > |\bar{A}\backslash\bar{s}\bar{A}|$, then there are a and b in A such that $\bar{a} = \overline{sb}$. Since A is finite, there are at most finitely many relations of the form so mentioned above which holds with s in F and a,b in A. Let $\overline{a_i} = \overline{s_ib_i}$ (i = 1, 2,..., m) be an enumeration of all of them. Then, there are $u_1, u_2, \ldots, u_m$ in S such that $a_iu_1\ldots u_i = s_ib_iu_1\ldots u_i$, $1 \leq i \leq m$. Let $A_0 = \{au_1\ldots u_m : a \text{ in } A\}$. Since S/R has right cancellation, we have $|\bar{A}| = |\bar{A}_0| \leq |A_0| \leq |A| = |\bar{A}|$ and hence $|A_0| = |A|$. Let $u = u_1u_2\ldots u_n$. If $\overline{au} = \overline{sbu}$, for some a, b in A and s in F, then $\bar{a} = \overline{sb}$ and hence $a = a_i$, $b = b_i$ and $s = s_i$, for some i = 1, 2, ..., m. Now, we have $au_1\ldots u_i = sbu_1\ldots u_i$ and thus $au = (au_1 \ldots u_i)(u_{i+1} \ldots u_m) = (sbu_1 \ldots u_i)(u_{i+1} \ldots u_m) = sbu$. Consequently, we have shown that, for any s in F and for any au, bu in A_0, $\overline{au} = \overline{sbu}$ implies $au = sbu$. Hence $|A_0 \setminus sA_0| = |\bar{A}_0 \setminus \bar{s}\bar{A}_0|$, for each s in F. But, $|\bar{A}_0 \setminus \bar{s}\bar{A}_0| = |\bar{A} \setminus \bar{s}\bar{A}|$. It follows that $|A_0 \setminus sA_0| < \varepsilon |A_0|$, for each s in F. However, the Sorenson's conjecture has not been proved, or disproved, yet; and it becomes a very interesting problem.

§4. Combinatorial Results.

In this section, first we show that one can get new amenable groups from any given amenable groups through the following processes: (1) by taking the subgroups of amenable groups; (2) by taking the quotient groups of amenable groups; (3) by taking the extension groups of amenable groups; (4) by taking the direct limit of amenable groups. Then we prove that finite groups and Abelian groups are amenable; and free groups on two, or more, generators are not amenable. This leads to the following unsolved problems stated by Day [5]: (i) whether every amenable group can be obtained from finite groups and Abelian groups through these four processes; and (ii) whether the family of all groups with no non-Abelian free subgroups is exactly the family of all amenable groups.

4.1. Proposition. *If a semigroup is both left and right amenable,* *then it is amenable.*

Proof. This has already been observed in Corollary 2.13.

4.2. Proposition. *A left [right] amenable group* G *is right [left] amenable; hence is amenable.*

Proof. Let μ be a left invariant mean on $m(G)$. Define a linear operator $T: m(G) \to m(G)$ by $Tf(x) = f(x^{-1})$, for each f in $m(G)$ and x in G. It follows easily from the definition of T that T is linear and $\|Tf\| = \|f\|$, for each f in $m(G)$. Let T^* be the adjoint operator of T and μ be a left invariant mean on $m(G)$. Observe that $T1 = 1$ and $Tf \geq 0$, if $f \geq 0$. Hence $T^*\mu(1) = 1$ and $T^*\mu(f) \geq 0$ if $f \geq 0$. Thus, $T^*\mu$ is a mean. Moreover, $T(\gamma_x f) = \ell_{x^{-1}} Tf$, for each f in $m(G)$ and each x in G. It follows that

$$T^*\mu(\gamma_x f) = \mu(T(\gamma_x f)) = \mu(\ell_{s^{-1}} Tf) = \mu(Tf) = T^*\mu(f),$$

for each x in G and each f in $m(G)$. Consequently, G is right amenable.

4.3. Proposition. *A homomorphic image of a left amenable semigroup is left amenable.*

Proof. Let $h: S \to T$ be a semigroup homomorphism from S onto T. Suppose that S is left amenable. Let T be represented as continuous affine maps from a compact convex set K in a locally convex space E into K itself. Then, $\{h(s):s \in S\}$ is a representation of S over K. Hence, there is a common fixed point k_0, say, for all $h(s)$, s in S. But h is onto. Thus, k_0 is a common fixed point of all t in T. By Theorem 3.6, T is left amenable.

4.4. Corollary. *Every factor group of an amenable group is amenable.*

Proof. The quotient map is a surjective (onto) homomorphism.

4.5. Proposition. *Every subgroup of an amenable group is amenable.*

Proof. Let H be a subgroup of an amenable group G. For each x in G, let $\bar{x}$ be an arbitrary but fixed element in the right coset H_x. Hence, for every x in G, there is a unique h_x in H such that $x = h_x\bar{x}$. Define a linear operator $T:m(H) \to m(G)$ by $Tf(x) = f(h_x)$, for each f in $m(H)$ and x in G. Since h_x is uniquely determined and f is bounded, Tf is well-defined and is in $m(G)$. Furthermore, for any f and g in $m(H)$ and for any scalar α and β,

$$\begin{aligned} T[\alpha f+\beta g](x) &= [\alpha f+\beta g](h_x) \\ &= \alpha f(h_x)+\beta g(h_x) \\ &= \alpha Tf(x)+\beta Tg(x) \\ &= [\alpha Tf+\beta Tg](x), \end{aligned}$$

for each x in G. Therefore, T is linear. It follows from

$$\|Tf\| = \sup\{|f(h_x)| : x \text{ is in } G\} < \|f\|$$

that T is bounded. Moreover, for any a in H and x in G, $ax = h_{ax}\overline{ax} = h_{ax}\bar{x}$ (since $\overline{ax} = \bar{x}$), and hence $ah_x = h_{ax}$. Consequently,

$$\begin{aligned} T(f-\ell_a f)(x) &= Tf(x) - T(\ell_a f)(x) \\ &= Tf(x) - f(ah_x) \\ &= Tf(x) - f(h_{ax}) \\ &= Tf(x) - Tf(ax) \\ &= [Tf - \ell_a Tf](x), \end{aligned}$$

for each f in m(H) and a in H. It follows that T maps $K_{m(H)}$ into $K_{m(G)}$. Since G is left amenable, by (3.6.7) of Theorem 3.6, we have

$$\sup\{h(x) : x \text{ in } H\} \geq \sup\{Th(x) : x \text{ in } G\} \geq 0,$$

for each h in $K_{m(H)}$ and hence H is left amenable.

4.6. Remark. Not every subsemigroup of a left amenable semigroup is left amenable. For example, let S be a non-left amenable semigroup. Let $S' = S \cup \{o\}$ with $so = os = o$, for all s in S'. Then, the mean μ defined by $\mu(f) = f(o)$, for each f in m(S) is a left invariant mean and S is not left amenable.

Furthermore, it is not even true that a subsemigroup of an amenable group is left amenable. (For the details, see [19].)

4.7. Proposition. *Let* H *be a normal subgroup of* G. *Then,* G *is amenable if and only if* H *and* G/H *are amenable.*

Proof. The necessity follows immediately from Proposition 4.5 and Corollary 4.4.

Conversely, suppose that H and G/H are amenable. Let $\{T_s : s \text{ in } G\}$ be a representation of G as continuous affine maps from a compact convex set K in a locally convex set E into K. Since H is amenable, the set K_0 of all k in K such that $T_h(k) = k$, for each h in H, is not empty. Also, K_0 is convex and is closed in K so that K_0 is a compact convex set in E. For each x in G, let $\bar{x}$ denote the coset xH in G/H. For each $\bar{x}$ in G/H, we define a continuous affine map $T_{\bar{x}}$ on K_0 by $T_{\bar{x}}(k) = T_x(k)$, for each k in K_0. If $\bar{x} = \bar{y}$, then $y^{-1}x$ is in H and hence $T_{y^{-1}x}(k) = k$, for each k in K_0. Thus,

$$T_{\bar{x}}(k) = T_x(k) = T_y(k) = T_{\bar{y}}(k),$$

for every k in K_0. Hence, $T_{\bar{x}}$ is well-defined. Also, since H is a normal subgroup of G, for each h in H and x in G, there is an s in H such that hx = xs. It follows that, for each x in G and each h in H,

$$T_h[T_{\bar{x}}(k)] = T_h[T_x(k)] = T_x[T_s(k)] = T_x(k) = T_{\bar{x}}(k),$$

for each k in K_0 and some s in H. Hence, $T_{\bar{x}}$ maps K_0 into itself. Consequently, $\{T_{\bar{x}}:\bar{x}$ in $G/H\}$ is a representation of G/H as continuous affine maps from K_0 into K_0. Hence, there is a k_0 in K_0 such that $T_x(k_0) = T_{\bar{x}}(k_0) = k_0$, for each x in G. By Theorem 3.6, G is left amenable and hence is amenable.

Let G_i, $i = 1, 2, 3$, be groups. If $\alpha:G_1 \to G_2$ and $\beta:G_2 \to G_3$ are group homomorphisms such that β is onto and $\alpha(G_1) = \mathrm{Ker}(\beta) = \beta^{-1}(e_3)$, where e_3 is the identity of G_3, then $G_1 \overset{\alpha}{\to} G_2 \overset{\beta}{\to} G_3$ is called an exact sequence. Given any exact sequence like above, the group G_2 is called an extension of G_1 by G_3. (See [24, p. 460].) Since $\mathrm{Ker}(\alpha)$ is a normal subgroup of G_1 (see [24, Theorem 23, p. 106]) and $\alpha(G_1) = \mathrm{Ker}(\beta)$, $G_1/\mathrm{Ker}(\alpha)$ is isomorphic to $\mathrm{Ker}(\beta)$ (see Theorem [24, Theorem 22, p. 105]). On the other hand, β is onto implies that $G_2/\mathrm{Ker}(\beta)$ is isomorphic to G_3. Consequently, if G_1 and G_3 are amenable, then Proposition 4.3 and Corollary 4.4 imply $\mathrm{Ker}(\beta)$ and $G_2/\mathrm{Ker}(\beta)$ are amenable. Hence, according to Proposition 4.7, G_2 is also amenable. Thus, we have shown that extensions of amenable groups are amenable.

4.8. <u>Proposition</u>. *Suppose that* $\{S_i\}_{i\in I}$ *is a family of subsemi-*

groups of semigroup S *such that (a)* $S = \bigcup\{S_i : i \in I\}$ *and (b) for all* i *and* j *in* I*, there is a* k *in* I *such that* $S_i \cup S_j \subseteq S_k$*. If, for each* i *in* I*,* S_i *is left amenable, then* S *is left amenable.*

Proof. Let S be represented as continuous affine maps from a convex compact set K in a locally convex space E into K itself. Suppose that each S_i is left amenable. Then, for each i, the set K_i of all common fixed points for S_i in K is not empty. Also, K_i is closed in K. By the property (b) we have, for any i and j in I there is an m in I such that $K_m \subseteq K_i \cap K_j$. Hence, the family $\{K_i : i \text{ in } I\}$ of closed subsets in K has finite intersection property. It follows from the compactness of K that $\bigcap\{K_i : i \in I\}$ is non-void. Consequently, S has a common fixed point and hence, by Theorem 3.6, S is left amenable.

4.9. Corollary. *A group* G *is amenable if and only if every finitely generated subgroup of* G *is amenable.*

Proof. By Proposition 4.5, the necessity is evident. The sufficiency follows from Proposition 4.8.

4.10. Proposition. *The full direct product of finitely many number of left amenable semigroups is left amenable.*

Proof. We prove only that the full direct product of two left amenable semigroups is left amenable. Then the rest of the proof follows from induction. Let S_1 and S_2 be left amenable semigroups. Suppose that μ_i is a left invariant mean on $m(S_i)$, $i = 1,2$. For each f in $m(S_1 \times S_2)$ and each fixed s in S_1, define f_s in $m(S_2)$ by $f_s(t) = f(s,t)$, for each t in S_2. Since f is bounded, $\|f_s\| \leq \|f\|$, for each s in S_1, and hence f_s is in $m(S_2)$. For each f in $m(S_1 \times S_2)$, define $f^{\sim}$ on S_1 by $f^{\sim}(s) = \mu_2(f_s)$, for each s in S_1. Since $|f^{\sim}(s)| \leq \|\mu_2\| \, \|f_s\| \leq \|f\|$, $f^{\sim}$ is in $m(S_1)$. Define a linear functional μ on $m(S_1 \times S_2)$ by $\mu(f) = \mu_1(f^{\sim})$, for each f in $m(S_1 \times S_2)$. If $f = g$ in $m(S_1 \times S_2)$, then $f_s = g_s$, for all s in S_1 and hence $f^{\sim} = g^{\sim}$. Thus, μ is well-defined. Observe that, for any f and g in $m(S_1 \times S_2)$ and any scalars α and β, $[\alpha f + \beta g]_s = \alpha f_s + \beta g_s$, for each s in S_1. It follows that $[\alpha f + \beta g]^{\sim} = \alpha f^{\sim} + \beta g^{\sim}$ and hence μ is linear. Moreover, if $f \geq 0$ is in $m(S_1 \times S_2)$, then $f_s \geq 0$ and hence $f^{\sim}(s) = \mu_2(f_s) \geq 0$, for each s in S_1. Thus, $\mu(f) \geq 0$ if $f \geq 0$. Also, $1_{S_1 \times S_2}{}^{\sim} = 1_{S_1}$ so that $\mu(1_{S_1 \times S_2}) = \mu_1(1_{S_1}) = 1$. By Proposition 1.2, μ is a mean on $m(S_1 \times S_2)$. Finally, to prove that μ is a left invariant mean, let (a,b) be in $S_1 \times S_2$ and f in $m(S_1 \times S_2)$. Then

$$(\ell_{(a,b)} f)_s(t) = f(as,at) = \ell_b(f_{as})(t),$$

for every (s,t) in $S_1 \times S_2$. Thus, it follows from the following equalities:

$$\begin{aligned}(\ell_{(a,b)}f)^{\sim}(s) &= \mu_2[(\ell_{(a,b)}f)_s] \\ &= \mu_2[\ell_b(f_{as})] \\ &= \mu_2[f_{as}] \\ &= f^{\sim}(as) \\ &= \ell_a(f^{\sim})(s),\end{aligned}$$

that

$$\begin{aligned}\mu[\ell_{(a,b)}f] &= \mu_1[(\ell_{(a,b)}f)^{\sim}] \\ &= \mu_1[\ell_a(f^{\sim})] \\ &= \mu_1[f^{\sim}] \\ &= \mu(f),\end{aligned}$$

for each f in $m(S_1 \times S_2)$ and (a,b) in $S_1 \times S_2$. Hence, $S_1 \times S_2$ is a left amenable semigroup.

4.11. Proposition. *The weak direct product of a family of amenable groups is amenable.*

Proof. Let $\{G_i\}_{i\varepsilon I}$ be a family of amenable groups. Recall that the weak direct product ΣG_i of $\{G_i\}_{i\varepsilon I}$ is the subgroup of the full direct product $\Pi_{i\varepsilon I} G_i$ which consists of all elements $(x_i)_{i\varepsilon I}$ such that $x_i = e_i$ for all but finitely many indices. Let Δ be the family of all non-void finite subsets of I. For each σ in

Δ, let $S_\sigma = \prod_{i\varepsilon\sigma} G_i$. Then, by Proposition 4.10, S_σ is amenable. Moreover, $\Sigma G_i = \bigcup\{S_\sigma : \sigma$ in $\Delta\}$ and, for any σ_1 and σ_2 in Δ, $S_{\sigma_1} \bigcup S_{\sigma_2} \subseteq S_{\sigma_1 \bigcup \sigma_2}$. Hence, by Proposition 4.8, ΣG_i is amenable.

4.12. Proposition. *The direct limit of amenable group is amenable.*

Proof. Let $\{G_i\}_{i\varepsilon I}$ be a family of amenable group which is indexed by a directed set $(I,\leq)$ such that (1) whenever $i \leq j$ in I, there is a group homomorphism $f_{ij}:G_i \to G_j$ such that $f_{jk} \circ f_{ij} = f_{ik}$ if $i \leq j$ and $j \leq k$; (2) for each i in I, f_{ii} is the identity map on G_i into itself. Let N be a subset of ΣG_i, the weak direct product of $\{G_i\}$, consisting of those $(x_i)_{i\varepsilon I}$ for which there is an index j (depending on $(x_i)_{i\varepsilon I}$) such that (a) $i \leq j$ whenever $x_i \neq e_i$ and (b) such that $\{f_{ij}(x_i) : i \leq j\} = \{e_j\}$, where e_i is the identity of G_i, for each i in I. First, we claim that N is a normal subgroup of ΣG_i. Let $(x_i)_{i\varepsilon I}$ and $(y_i)_{i\varepsilon I}$ be in N. Then there are ind ces j and k in I (depending on $(x_i)_{i\varepsilon I}$ and $(y_i)_{i\varepsilon I}$, respectively) such that $(x_i)_{i\varepsilon I}$ and $(y_i)_{i\varepsilon I}$ satisfy (a) and (b). Let m in I be such that $j \leq m$ and $k \leq m$. If $i \leq m$ such that $i \leq j$ or $i \leq k$, then we have $f_{im}(x_i) = f_{jm}(f_{ij}(x_i)) = f_{jm}(e_j) = e_m$ or $f_{im}(y_i^{-1}) = [f_{km}(f_{ik}(y_i))]^{-1} = [f_{km}(e_k)]^{-1} = e_m$. If $i \leq m$ such that either $i \leq j$ or $i \leq k$ does not hold, then $x_i = e_i$ or $y_i = e_i$. Hence, we have $f_{im}(x_i y_i^{-1}) = e_m$, whenever $i \leq m$. If $x_i y_i^{-1} \neq e_i$, then $x_i \neq e_i$ or $y_i \neq e_i$ and hence

$i \leq j$ or $i \leq k$. Thus, in either case, $i \leq m$. Therefore, we have shown that N is a subgroup of ΣG_i. Now, to prove that N is normal, let $(x_i)_{i\varepsilon I}$ be in N and (y_i) be in ΣG_i. Suppose j in I is such that $(x_i)_{i\varepsilon I}$ satisfies (a) and (b). If $y_i^{-1}x_iy_i \neq e_i$, then $x_i \neq e_i$ and hence $i \leq j$. If $i \leq j$, then we have $f_{ij}(y_i^{-1}x_iy_i) = [f_{ij}(y_i)]^{-1}f_{ij}(y_i) = e_j$. Hence, N is normal. Since the direct limit of $\{G_i\}_{i\varepsilon I}$ is the factor group $G_\infty = \Sigma G_i/N$, (see [21, p. 10]), by Proposition 4.10 and Corollary 4.4, G_∞ is amenable.

We have shown in Proposition 4.5 and Remark 4.6 that every subgroup of an amenable group is amenable but not every subsemigroup of a left amenable semigroup is left amenable. In the following proposition, we give a sufficient condition for a subsemigroup of a left amenable semigroup to be left amenable.

4.13. <u>Proposition</u>. *Suppose that* H *is a subsemigroup of* S. *If* μ *is a left invariant mean on* $m(S)$ *such that* $\mu(1_H) > 0$, *then* H *is left amenable.*

<u>Proof</u>. For each f in $m(H)$ we define a linear operator $T:m(H) \to m(S)$ by $Tf(s) = f(s)$ if s in H and $Tf(s) = 0$ otherwise. The linearity of T follows easily from the definition. Also, $\|Tf\| \leq \|f\|$, for each f in $m(H)$, hence T is bounded by 1. Define μ_0 on $m(H)$ by

$$\mu_0(f) = \mu(Tf)/\mu(1_H),$$

for each f in $m(H)$. If $f \geq 0$ in $m(H)$, then $Tf \geq 0$ and hence $\mu_0(f) \geq 0$. Moreover, $\mu_0(1_H) = 1$. Hence, by Proposition 1.2, μ_0 is a mean on $m(H)$. To prove that μ_0 is left invariant, we claim that $\mu[T(\ell_s f) - \ell_s(Tf)] = 0$, for each s in H and f in $m(H)$. Then, if our assertion is valid, $\mu[T(\ell_s f)] = \mu[\ell_s(Tf)] = \mu(Tf)$ and hence $\mu_0(\ell_s f) = \mu_0(f)$, for each s in H and f in $m(H)$. Now, to prove our assertion, let f be in $m(H)$ and s in H. Define $g = T(\ell_s f) - \ell_s(Tf)$. For the same s, let $E = s^{-1}H \cap [S \setminus H]$. For every f and h in $m(S)$, define fh in $m(S)$ by $fh(t) = f(t)h(t)$, for each t in S. Then $g1_E(t) = g(t)$, if t in E and $g(t) = 0$ otherwise. For each t in S, there is at most one element in $\{s^i t : i = 1, 2, \ldots\}$ that belongs to E. Since if i is the smallest integer such that $s^i t$ is in $E = s^{-1}H \cap (S \setminus H)$, $s^{i+k}t$ is not in $S \setminus H$ and hence is not in E, for any integer $k \geq 1$. Therefore, for every integer $n > 0$, we have $\sum_{i=1}^n \ell_{s^i} 1_E(t) \leq 1$, for all t in S. Since μ is left invariant,

$$\begin{aligned} n\mu(1_E) &= \textstyle\sum_{i=1}^n \mu(\ell_{s^i} 1_E) \\ &= \mu[\textstyle\sum_{i=1}^n \ell_{s^i} 1_E] \\ &\leq \mu(1) \\ &= 1, \end{aligned}$$

for all integer $n > 0$. Hence $\mu(1_E) = 0$. Since $g = g1_E$, we have $g \leq ||g|| 1_E$. It follows that $|\mu(g)| \leq \mu(||g|| 1_E) = ||g|| \mu(1_E) = 0$. This proves our assertion and hence completes the proof.

4.14. Proposition. *Let* S *be a left amenable semigroup. Then the family of all right ideals of* S *has finite intersection property.*

Proof. Assume the contrary that there are two right ideals H and R such that $H \cap R = \phi$. Let a be in H and b be in R. Define $f = 1_H - 1_R$. Then $\ell_a f = 1$ and $\ell_b f = -1$. But $h = \ell_b f - \ell_a f$ is in K and $\sup\{h(s) : s \in S\} = -2$. Hence S is not left amenable. This completes the proof.

4.15. Definition. A subset H of a semigroup S is called left thick if for any finite subset F of S, there is an s in S such that $Fs \subseteq H$. If, addition, H is a subsemigroup of S, then H is called a left thick subsemigroup.

This concept of left thick subsets is due to Mitchell [25]. Let H be a left ideal of S and F be a finite subset of S. Then, $Fs \subseteq H$, for any s in H. Hence, every left ideal is left thick. In a left amenable semigroup S, every right ideal is left thick. Suppose that H is a right ideal of S. For any finite set F of S, by Proposition 4.14, we have $H \cap (\bigcap_{s \in F} sS) \neq \phi$. It follows that $\bigcap_{s \in F} s^{-1}H \neq \phi$, where

$s^{-1}H = \{t\varepsilon S : st \ \varepsilon \ H\}$. Consequently, there is a t in $\bigcap_{s\varepsilon F} s^{-1}H$ such that $Ft \subseteq H$. Hence, H is left thick.

Suppose that H is a left thick subset of S. For each finite subset F of S, by definition, there is an s in S such that $Fs \subseteq H$. If s is not in H, then there is a t in S such that $(Fs \bigcup \{s\})t \subseteq H$. Consequently, there is an st in H such that $F(st) \subseteq H$. Hence, for any finite subset F of S, there is a s in H such that $Fs \subseteq H$.

4.16. <u>Proposition</u>. *Let* T *be a left thick subsemigroup of* S. *If* S *is left amenable, then there is a left invariant mean* μ *on* $m(S)$ *such that* $\mu(1_T) = 1$ *and hence* T *is left amenable.*

Conversely, if T *is left amenable, then* S *is left amenable also.*

<u>Proof</u>. Suppose that S is left amenable. Then, by Theorem 3.6, there is a net $\{\phi_\alpha\}$ of finite means such that $\lim_\alpha \| 1_s\phi_\alpha - \phi_\alpha \| = 0$, for each s in S. For each s in S, there is an s_α in S such that $F_\alpha s_\alpha \subseteq T$, where $F_\alpha = \{s\varepsilon S : \phi_\alpha(s) \neq 0\}$. Then, each $\phi_\alpha 1_{s_\alpha}$ is a finite mean with support in T. Since $\ell_1(S)$ is a Banach algebra, we have, for each s in S,

$$\begin{aligned} \| 1_s(\phi_\alpha 1_{s_\alpha}) - \phi_\alpha 1_{s_\alpha} \| &= \| (1_s\phi_\alpha)1_{s_\alpha} - \phi_\alpha 1_{s_\alpha} \| \\ &= \| (1_s\phi_\alpha - \phi_\alpha)1_{s_\alpha} \| \\ &\leq \| 1_s\phi_\alpha - \phi_\alpha \| \, \| 1_{s_\alpha} \| \end{aligned}$$

$$= \| 1_s\phi_\alpha - \phi_\alpha \| .$$

Hence, $\{\phi_\alpha 1_{s_\alpha}\}$ is norm-convergent to left invariance also. By the ω^*-compactness of the set of all means, there is a ω^*-cluster point, say μ, of $\{\phi_\alpha 1_{s_\alpha}\}$. Moreover, $\mu(1_T) = 1$, since each $\phi_\alpha 1_{s_\alpha}$ has support in T. It follows from Corollary 2.13 that μ is a left invariant mean on $m(S)$. This completes the proof.

Now, suppose that T is left amenable. Recall that for any finite subset F of S, we can always find a t in T such that $Tt \subseteq T$. Let S be represented as continuous affine map from a compact convex set K in a locally convex space into itself. Since T is left amenable. There is a $k \in K$ such that $t(k) = k$, for each t in T. Now, for each s in S, there is a t in T such that st is in T. Hence,

$$s(k) = s(t(k)) = st(k) = k.$$

Thus, S has a common fixed point and is left amenable. This completes the proof.

4.17. <u>Proposition</u>. *A finite semigroup* S *is left amenable if and only if it has a unique minimal right ideal* R. *Then* R *is the union of disjoint minimal left ideals* $L_1, L_2, \ldots, L_n$ *of* S *such that each* L_i *is a group and is isomorphis to each other. Moreover,* $\mu_i = \frac{1}{|L_i|} \sum_{s \in L_i} 1_s$, $1 \leq i \leq n$, *is a left invariant mean and each left invariant mean is a convex combination of the* μ_i.

<u>Proof</u>. Suppose, first, that S is left amenable. Let $\mathcal{C}$ be the family of all right ideals of S. Since S is finite, $\mathcal{C}$ is finite. Since the intersection of two right ideals is a right ideal, it follows from Proposition 4.14 that $\bigcap\mathcal{C}$ is non-void and is a right ideal. By definition, $\bigcap\mathcal{C}$ is the unique minimal right ideal.

Conversely, let R be the unique minimal right ideal. For each s in S, sR is also a right ideal. If $sR\cap R = \phi$, then there is a minimal right ideal H, say, such that $H \subseteq sR$ and $H\cap R = \phi$. But since R is unique, this can not be. Hence, $sR\cap R \neq \phi$. Since $sR\cap R$ is a right ideal contained in R, we have $sR\cap R = R$ and hence $|R\setminus sR| = 0$, for each s in S. Thus S satisfies the strong Følner's condition and is left amenable.

Now, to prove the second part of the proposition, we claim that the union of all disjoint minimal left [right] ideals of S is a right [left] ideal. To see this, let $L_1, L_2, \ldots L_n$ be all the disjoint minimal left ideals of S and $H = \bigcup_{i=1}^{n} L_i$. For each s in S, $L_i s$ is a left ideal, $1 \leq i \leq n$. Hence $L_i s \supseteq L_j$, for some j, $1 \leq j \leq n$. Let $L = \{t \varepsilon L_i : ts \ \varepsilon\ L_j\} \subseteq L_i$. Then, for any x in S, (xt) is in L_i, for each t in L, and $(xt)s = x(ts)$ is in L_j. Thus, xt is in L and $xL \subseteq L$. But L_i is minimal. It follows that $L = L_i$ and $L_i s = L_j$. Consequently, $Hs \subseteq \bigcup_{i=1}^{n} L_i s \subseteq \bigcup_{j=1}^{n} L_j = H$ and H is a right ideal.

Similarly, we can prove that the union of all disjoint minimal right ideals is a left ideal. Now, let R be the unique minimal right ideal of S. By the above claim, R is also a left ideal. Let $L_i, \ldots L_n$, and H be defined as above. Then H is a right ideal and hence $H \supseteq R$. For each i, $1 \le i \le n$, RL_i is a left ideal, since R is a left ideal, and $RL_i \subseteq L_i$. It follows that $RL_i = L_i$. But, on the other hand, since R is a right ideal, we have $L_i = RL_i \subseteq R$, for each i, $1 \le i \le n$. Consequently, $H = \bigcup_{i=1}^{n} L_i \subseteq R$ and thus $H = R$. To prove that each L_i is a group, we observe that every cancellative finite semigroup is a group. (See [17, Theorem 9.16, p. 99].) We now prove that each L_i satisfies the cancellation laws. For each s in R, $sR \subseteq R$. But sR is a right ideal. Thus $sR = R$ and R has left cancellation. Hence, $L_i \subseteq R$ has left cancellation, for each $1 \le i \le n$. On the other hand, for each s in L_i, $1 \le i \le n$, $L_i s \subseteq L_i$ and $L_i s$ is a left ideal. Therefore, $L_i s = L_i$, $1 \le i \le n$. Thus, L_i has right cancellation also, $1 \le i \le n$. It follows that L_i is a group, $1 \le i \le n$. Let e_i be the identity of L_i, $1 \le i \le n$. Let i and j, $1 \le i$, $j \le n$, be arbitrary. Since $L_i e_j = L_j$, the mapping $h: t \to te_j$ is a one-to-one map from L_i onto L_j. For any s and t in L_i, we have

$$h(st) = (st)e_j = s(te_j) = s(e_j(te_j)) = (se_j)(te_j) = h(s)h(t)$$

and h is an isomorphism. Consequently, L_i is isomorphic to L_j for any $1 \leq i, j \leq n$.

Now, to prove that $\mu_i = \frac{1}{|L_i|} \sum_{s\epsilon L_i} 1_s$, $1 \leq i \leq n$, is a left invariant mean, we first claim that $tL_i = L_i$, for each t in S and each i, $1 \leq i \leq n$. Then, $\mu_i(\ell_t f) = \frac{1}{|L_i|} \sum_{s\epsilon L_i} f(ts) = \frac{1}{|L_i|} \sum_{ts\epsilon tL_i} f(ts) = \mu_i(f)$. To prove our claim, observe that $tL_i \subseteq L_i$, for each i. If $ta = tb$, for some a and b in L_i, then $ta = t(e_i a) = t(e_i b) = tb$. But te_i is in L_i and hence $(te_i)a = (te_i)b$ implies that $a = b$. This proves that $tL_i = L_i$, for each t in S.

Finally, we prove that each left invariant mean of $m(S)$ is a convex combination of the μ_i. Let μ be a left invariant mean of $m(S)$. Since S is finite, each mean is a finite mean and $\mu = \sum_{s\epsilon S}\mu(s)1_s$. By Proposition 4.16, $\mu(1_R) = \sum_{s\epsilon R}\mu(s) = 1$ and thus $\mu = \sum_{s\epsilon R}\mu(s)1_s$. Since $R = \bigcup_{i=1}^{n} L_i$, we have $\mu = \sum_{i=1}^{n} \sum_{s\epsilon L_i}\mu(s)1_s$. For each $1 \leq i \leq n$, observe that $\ell_s 1_s \geq 1_{e_i}$ and $\ell_{s^{-1}} 1_{e_i} \geq 1_s$, for each s in L_i, where s^{-1} is the inverse of s in L_i (since $\ell_s 1_s(e_i) = 1$ and $\ell_{s^{-1}} 1_{e_i}(s) = 1$, the inequalities hold when x and t in S are such that $sx = x$ and $s^{-1}t = e_i$, respectively). It follows that $\mu(1_s) = \mu(\ell_s 1_s) \geq \mu(1_{e_i})$ and $\mu(1_{e_i}) = \mu(\ell_{s^{-1}} 1_{e_i}) \geq \mu(1_s)$ and μ takes a constant value on each L_i. Thus, we have $\mu = \sum_{i=1}^{n}(\mu(1_{e_i})\sum_{s\epsilon L_i} 1_{s_i})$ $= \sum_{i=1}^{n}\mu(1_{e_i})|L_i|\mu_i$. Since μ is a mean, we have

$$\begin{aligned} 1 &= \sum_{s\varepsilon S}\sum_{i=1}^{n}\mu(1_{e_i})|L_i|\mu_i(s) \\ &= \sum_{i=1}^{n}\mu(1_{e_i})|L_i|\sum_{s\varepsilon S}\mu_i(s) \\ &= \sum_{i=1}^{n}\mu(1_{e_i})|L_i|. \end{aligned}$$

Since $\mu(1_{e_i})|L_i| \geq 0$, $1 \leq i \leq n$, μ is a convex combination of the μ_i.

4.18. Corollary. *A finite semigroup* S *is amenable if and only if it has a unique minimal left ideal* L *and a unique minimal right ideal* R *such that* R = L. *Then there is precisely one two-sided invariant mean on* S.

Proof. The first part of the corollary is clear from Proposition 4.18. The unqiueness of the invariant mean follows from the fact the each left [right] invariant mean is of the form

$\frac{1}{N}\sum_{s\varepsilon L}1_s$ $\quad$ $[\frac{1}{N}\sum_{s\varepsilon R}1_s]$, where $N = |R| = |L|$.

4.19. Corollary. *A finite left cancellative semigroup* S *is left amenable. If, in addition,* S *is a group, then it has a unique invariant mean.*

Proof. Since a left cancellative finite semigroup S has only one right ideal, namely S itself, by Proposition 4.17, S is left amenable. If S is a group, then it has only one left ideal S and one right ideal S. By Corollary 4.18, the invariant mean is unique.

4.20. <u>Proposition</u>. *Free semigroup [group] on two, or more generators is not amenable.*

<u>Proof</u>. Let S be a free semigroup generated by a and b. Define a bounded real-valued function f on S by f(s) = 1 if s is a word begins with a and f(s) = 0 otherwise. Then, the function $h = \ell_b f - \ell_a f = (\ell_b f - f) - (\ell_a f - f)$ is in K. Hence $\sup\{h(s) : s \text{ in } S\} = -1 < 0$ and, by (3.6.7) of Theorem 3.6, S is not left amenable.

4.21. <u>Corollary</u>. *The full direct product of amenable group is not necessary amenable.*

<u>Proof</u>. Since the free group on two generators is isomorphic to a subgroup of the product group of a family of finite groups (see [23, Corollary 8.21]) and finite groups are amenable, it follows from Proposition 4.20 that there is a family of amenable groups whose full direct product contains a non-amenable subgroup. By Proposition 4.5, this full direct product is not amenable.

One can show that Abelian semigroups are amenable by an easy application of the Markov-Kakutani fixed point theorem [7, Theorem 6, p. 456], or by showing that Abelian semigroups satisfy the Dixmier criterion ((3.6.7) of Theorem 3.6) as given in [17, Theorem 17.5, p. 231]. However, we choose to give a new proof of this fact by showing that Abelian semigroups satisfy the strong Følner's condition, i.e. given any finite subset F and any $\varepsilon > 0$, there is a finite set A such that $|A \setminus sA| < \varepsilon |A|$, for each s in F. Our proof is longer but we find that it has some merit in the fact that our proof will yield an explicit method of constructing the set A in the strong Følner's condition, and thus also a method of constructing a net of finite means converging to left invariance in norm (see Proposition 3.15).

We first prove the following two lemmas.

4.22. <u>Lemma</u>. *Let* A *be a finite subset of a semigroup* S. *Suppose* s *is in* S. *Then a necessary and sufficient condition for* $|sA| = |A|$ *is that* $sa = sb$ *implies* $a = b$, *for* a *and* b *in* A. *Furthermore, if* $|sA| = |A|$, *then* $|sB| = |B|$ *for all subset* B *of* A.

<u>Proof</u>. We observe that $|sA| \leq |A|$ is always true. Then $|sA| < |A|$ if and only if there are a and b in A such that $a \neq b$ and $sa = sb$. The Lemma is now clear.

4.23. <u>Lemma</u>. *Let* A *be a finite subset of a semigroup* S. *If* s *is an element in* S *and* k *is a positive integer such that*

(4.23.1) $\quad |A| = |sA| = \ldots = |s^{k-1}A|$; *and*

(4.23.2) $\quad s^j(A \setminus sA) \subseteq sA \cap A$, for $1 \leq j \leq k-1$, *then we have*

(4.23.3) $\quad k|A\setminus sA| \leq |A|.$

<u>Proof</u>. Let $B = A\setminus sA$. First we claim that $s^iB \cap s^jB = \phi$, for $0 \leq i, j \leq k-1$ and $i \neq j$. (Note that $s^0B = B$.) Since $B = A\setminus sA$, we have $B \cap sA = \phi$. By (4.32.2), it follows that $s^iB \subseteq sA \cap A \subseteq sA$, for $1 \leq i \leq k-1$, Hence, $B \cap s^iB \subseteq B \cap sA = \phi$. Now, suppose that $s^iB \cap s^jB \neq \phi$, for some $i \neq j$ and $1 \leq i$, $j \leq k-1$. Then there are a and b in B such that $s^ia = s^jb$. Assume that $i < j$. Then we have $s^jb = s^i(s^{j-i}b)$. Since $1 \leq i \leq j \leq k-1$, by (4.23.2), $s^{j-i}b \in s^{j-i}B \subseteq sA \cap A \subseteq A$. Also, a is in $B \subseteq A$; hence a is in A. Since $|s^iA| = |A|$ and $s^ia = s^jb = s^i(s^{j-i}b)$, by Lemma 4.22, we have $a = s^{j-i}b$. This contradicts that $B \cap s^{j-i}B = \phi$. Thus, our assertion is proven. To establish the inequality (4.23.3), we observe that

$$B \cup sB \cup \ldots \cup s^{k-1}B \subseteq A,$$

since $B \subseteq A$ and $s^iB \subseteq sA \cap A \subseteq A$, for $1 \leq i \leq k-1$. Hence, by our assertion, $|B| + |sB| + \ldots + |s^{k-1}B| \leq |A|$. Since $|A\setminus sA| = |B| = |s^iB|$, $1 \leq i \leq k-1$, we have $k|A\setminus sA| \leq |A|$. This finishes the proof.

Let S be an Abelian semigroup. For each finite subset $B = \{b_1, b_2, \ldots, b_n\}$ in S and each integer k, we denote by B^k the finite subset of S consisting of elements in the form

$b_1^{i_1}b_2^{i_2}\dots b_n^{i_n}$, where $0 \leq i_p \leq k$, $1 \leq p \leq n$ and $\sum_{p=1}^{n} i_p \geq 1$. Here we use the convention that $ab^0 = a = b^0a$ for a and b in S. Also, we want to point out that the set B^k here does not represent the usual set $\{b_1b_2\dots b_k : b_i \in B,\ i = 1, 2, \dots, k\}$.

4.24. Theorem. *Every Abelian semigroup satisfies the strong Følner's condition; and hence is amenable.*

Proof. Let S be an Abelian semigroup. Let $F = \{s_1, s_2, \dots, s_n\}$ be an arbitrary finite subset of S and $\varepsilon > 0$ be given. Then there is a positive integer k such that $0 < \frac{1}{k} < \varepsilon$. Define $A_0 = F^k$. Then, for each j, $1 \leq j \leq n$,

$$A_0 \setminus s_jA_0 \subseteq (F\setminus\{s_j\})^k.$$

Since for any m, $1 \leq m \leq k$, $s_j^m(F\setminus\{s_j\})^k \subseteq F^k = A_0$, where $j = 1, 2, \dots, n$, we have $s_j^m(F\setminus\{s_j\})^k \subseteq s_jA_0$, for all $m = 1, 2, \dots, k-1$. Thus, for all $j = 1, 2, \dots, n$ and all $m = 1, 2, \dots, k-1$,

$$s_j^m(A_0\setminus s_jA_0) \subseteq s_jA_0 \cap A_0.$$

If $|A_0| = |s_j^mA_0|$, for all $1 \leq j \leq n$ and $1 \leq m < k$, then it follows from Lemma 4.23 that

$$|A_0 \setminus s_jA_0| \leq \frac{1}{k}|A_0| < \varepsilon|A_0|,$$

for all $1 \leq j \leq n$. However, if S is not cancellative, then this need not be the case. If $|s_j^mA_0| < |A_0|$, for some j, $1 \leq j \leq n$, and some m, $1 \leq m < k$, then we shall modify the set A_0 to have the desired properties. Let $u = s_1 s_2 \ldots s_n$. By the commutativity of S, we have $u^p = s_1^p \ldots s_n^p$, for $p = 1, 2, \ldots$. Since A_0 is finite, in fact $|A_0| \leq n^{k+1}$, $\{|A_0 u^p|\}_{p=1}^{\infty}$ is a non-increasing sequence of integers bounded above by n^{k+1} and below by 1. Hence, there is an integer p, say, such that

$$|A_0 u^p| = |A_0 u^{p+1}| = \ldots = |A_0 u^{p+k}|.$$

First, we claim that $|s_j^m A_0 u^p| = |A_0 u^p|$, for all $1 \leq j \leq n$ and $1 \leq m < k$. We observe that

$$\begin{aligned}|A_0 u^p| &\geq |s_j^m A_0 u^p| \geq |(s_1^m \ldots s_{j-1}^m)(s_{j+1}^m \ldots s_n^m) s_j^m A_0 u^p| \\ &= |u^m A_0 u^p| = |A_0 u^{p+m}| = |A_0 u^p|,\end{aligned}$$

for $1 \leq j \leq n$ and $1 \leq m < k$. Hence $|s_j^m A_0 u^p| = |A_0 u^p|$, for all j, $1 \leq j \leq n$, and m, $1 \leq m < k$. Let $A = A_0 u^p$. For each j, $1 \leq j \leq n$, we have

$$A \setminus s_j A = A_0 u^p \setminus s_j A_0 u^p \subseteq (A_0 \setminus s_j A_0) u^p.$$

Hence, when $1 \leq m \leq k-1$ and $1 \leq j \leq n$,

$$\begin{aligned}s_j^m (A \setminus s_j A) &\subseteq s_j^m ((A_0 \setminus s_j A_0) u^p) \\ &= (s_j^m (A_0 \setminus s_j A_0)) u^p \\ &\subseteq (s_j A_0 \cap A_0) u^p \\ &\subseteq s_j A_0 u^p \cap A_0 u^p \\ &= s_j A \cap A.\end{aligned}$$

By Lemma 4.23, we have $|A \setminus s_j A| \leq \frac{1}{k} |A| < \varepsilon |A|$, for all $j = 1, 2, \ldots n$. This completes the proof.

4.25. Remark. In general, one does not have much control on the size of the set A in the strong Følner's condition. However, if S is infinite and cancellative, then we can choose A to be arbitrarily large. To be more precise, given any $\varepsilon > 0$ and disjoint finite sets F and G, we can find a set A such that (i) $|A \setminus sA| < \varepsilon |A|$, for each s in F, and (ii) $A \supseteq F \cup G \supset G$. To see this, note that if S is cancellative then, for each finite set $F \subseteq S$ and each $\varepsilon > 0$, the set $A = F^k$, where $0 < \frac{1}{k} < \varepsilon$, satisfies that $|A| = |s^m A|$, for each s in F and $1 \leq m \leq k-1$. Now, the above claim can be easily accomplished by taking $A = (F \cup G)^k$.

4.26. Corollary. *Every solvable group is amenable.*

Proof. The corollary follows immediately from Proposition 4.16 and Theorem 4.24.

CHAPTER III <u>ERGODIC THEORY</u>

Ergodic theory is an outgrowth of a problem in statistical mechanics and Hamiltonian dynamics. A mechanical system is said to be "ergodic" if the time averages of its certain physical quantity converges to a constant as the time interval gets longer and longer. The physical assumption made on the system to ensure it to be ergodic is known as the "ergodic hypothesis". The mean ergodic theorem was first investigated by J. von Neumann [29]. It is an operator generalization of a very simple phenomenon: If α is a complex number, then the arithematic means $s_n(\alpha) = n^{-1}\sum_{i=1}^{n}\alpha^i$ converges when $|\alpha| \leq 1$; converges to 0 when $|\alpha| \lneq 1$; and diverges when $|\alpha| > 1$. J. von Neumann generalized this for one parameter unitary groups in Hilbert spaces. Next, Riesz [27] and Yosida [30] proved that if T is a bounded linear operator from a reflexive Banach space B into itself with $||T|| \leq 1$, then (1) the sequence $\{A_n\}$, where $A_n = n^{-1}\sum_{i=1}^{n}T^i$, converges strongly to a projection P; (2) $PT = TP = P$; (3) the range of P is the set of all fixed points of T and the null space of P is a closed linear subspace spanned by all elements $b - Tb$, b in B. Note that $S = \{T^i : i = 1, 2, \ldots\}$ forms an Abelian semigroup under functional composition. Furthermore, $||T^i|| \leq ||T||^i \leq 1$, for $i = 1, 2, \ldots$. In this thesis, we are interested in the case when S is a set of bounded linear operators from a Banach space

B into itself with $\sup\{\|s\|: s$ in $S\} < \infty$ such that S forms a semigroup under functional composition. We call S a bounded operator semigroup of B. For each b in B, the set $O(b) = \{s(b): s$ in $S\}$ is called the orbit of S. A linear operator A on B is called an average of S if, for each b in B, $A(b)$ is in $C\ell C_0 O(b)$, the uniform closure of the convex hull of $O(b)$; and A is called a finite average of S if $A(b)$, for each b in B, is in $C_0 O(b)$, the convex hull of $O(b)$.

1.1. Definition. A bounded operator semigroup S over a Banach space B is said to be weakly, strongly, or uniformly ergodic under a net $\{A_n\}$ of averages of S if, for each s in S, (use I for the identity operator on B)

(1.1.1) [weak] $\lim_n \beta[A_n(s-I)(b)] = 0 = \lim_n \beta[(s-I)A_n(b)]$;

for all β in B^* and b in B,

(1.1.2) [strong] $\lim_n \|A_n(s-I)(b)\| = 0 = \lim_n \|(s-I)A_n(b)\|$,

for each b in B;

(1.1.3) [uniform] $\lim_n \|A_n(s-I)\| = 0 = \lim_n \|(s-I)A_n\|$, respectively.

As an analog to Riesz and Yosida's results, we expect that $\{A_n\}$ converges to a projection P; and the range space of P is the set of all common fixed points of all s in S, and the null space of P is the closed subsapce of B which is spanned by the set $\{(s-I)(b): s$ in S and b in $B\}$. Hence, we define the following subspaces of B:

(i) $\mathcal{C}$ is the set of all common fixed points of all s in S;

(ii) K is the linear span of the set $\{(s-I)(b):s \in S$ and $b \in B\}$; and $C\ell(K)$ is the uniform closure of K in B.

Since the closure of a vector subspace is a vector subspace, $C\ell(K)$ is a closed vector subspace. Furthermore, the set of all common fixed points is closed and it follows from the following equalities:

$$s(\alpha a-\beta b) = \alpha s(a) - \beta s(b) = \alpha a-\beta b,$$

for each s in S, all a,b in S and all scalars α and β, that $\mathcal{C}$ is a closed vector subspace.

1.2. Lemma. *Let* S *be a bounded operator semigroup over a Banach space* B. *If* S *is weakly ergodic under a net* $\{A_n\}$ *of averages of* S, *then*

(1.2.1) $A_n(b) = b$, *for all* b *in* $\mathcal{C}$ *and each* n; *hence* $\lim_n A_n(b) = b$;

(1.2.2) $\omega\text{-}\lim_n A_n(b) = 0$ *if and only if* b *is in* K.

Proof. Suppose b is in $\mathcal{C}$. Then $C\ell C_0 O(b) = \{b\}$. Since $A_n(b)$ is in $C\ell C_0 O(b)$, for each n, we have $A_n(b) = b$. Hence, (1.2.1) holds.

For (1.2.2), we start with the sufficiency. Suppose that b_0 is in $C\ell(K)$. Since, for each b in B, $\omega\text{-}\lim_n A_n(s-I)b = 0$, for each s in S, we have $\omega\text{-}\lim_n A_n b = 0$, if b is in

K. Let $\{b_\alpha\}$ be a net in K such that $\lim_\alpha \|b_\alpha - b_0\| = 0$. Since S is bounded, $M = \sup\{\|s\|: s \text{ in } S\} < \infty$ and $\|A_n\| \leq M$, for each n. Hence, $\sup_n \|A_n\| \leq M$. Let β in B^* be arbitrary. For each $\varepsilon > 0$, there is an $\alpha(\varepsilon)$ such that $\|b_\alpha - b_0\| < \varepsilon/2M\|\beta\|$. Fix an $\alpha > \alpha(\varepsilon)$. Since $\lim_n \beta[A_n(b_\alpha)] = 0$, there is an $n(\alpha,\varepsilon)$ such that $\beta[A_n(b_\alpha)] < \varepsilon/2$ whenever $n > n(\alpha,\varepsilon)$. Thus, whenever $n > n(\alpha,\varepsilon)$.

$$\begin{aligned} |\beta(A_n(b_0))| &= |\beta(A_n(b_0 - b_\alpha) + A_n(b_\alpha))| \\ &= |\beta(A_n(b_0 - b_\alpha)) + \beta(A_n(b_\alpha))| \\ &\leq \|\beta\|\,\|A_n\|\,\|b_0 - b_\alpha\| + \varepsilon/2 \\ &< \varepsilon. \end{aligned}$$

Hence, $\lim_n \beta[A_n(b_0)] = 0$, for any arbitrary β in B^*, and this implies that $\omega\text{-}\lim_n A_n(b_0) = 0$.

Conversely, suppose b is not in $C\ell(K)$. Then $b \neq 0$ and $b + C\ell(K)$ is a closed affine subspace of B. For each s in S, $s(b) = b - (b - s(b))$ is in $b + C\ell(K)$. Since affine subspaces are convex and $b + C\ell(K)$ is closed, we have $C\ell C_0 O(b) \subseteq b + C\ell(K)$. Consequently, $\{A_n(b)\} \subseteq b + C\ell(K)$. But 0 is not in $b + C\ell(K)$ and, by Proposition I.1.7, $b + C\ell(K)$ is weakly closed. Therefore, $\{A_n(b)\}$ will not converge to 0 weakly. Thus, if $\omega\text{-}\lim_n A_n(b) = 0$, then b is in $C\ell(K)$.

Hereafter, let S be a bounded operator semigroup over a Banach space B. Also, let $F = \not{C} + C\ell(K)$, i.e., $F = \{c+k : c \in \not{C}$ and $k \in C\ell(K)\}$.

1.3. <u>Theorem</u>. *If* S *is weakly ergodic under a net* $\{A_n\}$ *of averages of* S, *then*

(1.3.1) $\not{C} \cap C\ell(K) = \{0\}$; *and hence* $F = \not{C} \oplus C\ell(K)$ *i.e.,* F *is the direct sum of* $\not{C}$ *and* $C\ell(K)$;

(1.3.2) $\omega\text{-}\lim_n A_n(b)$ *exists if and only if* b *is in* F;

(1.3.3) *if* $P(b) = \omega\text{-}\lim_n A_n(b)$, *for each* b *in* F, *then* P *is linear and* $||P|| \leq \sup_n ||A_n|| \leq \sup\{||s|| : s \text{ in } S\}$;

(1.3.4) *for each* b *in* F *and* s *in* S, *we have* $Ps(b) = sP(b) = P^2(b) = P(b)$;

(1.3.5) P *is a projection of* F *onto* $\not{C}$ *along* $C\ell(K)$; *that is,* $P(F) = \not{C}$ *and* $P(C\ell(K)) = \{0\}$;

(1.3.6) *for each* b *in* F, $C\ell C_0 O(b) \cap \not{C} = \{P(b)\}$;

(1.3.7) F *is closed.*

<u>Proof</u>. We first prove the condition (1.3.1). Since $\not{C}$ and $C\ell(K)$ are vector subspaces, 0 is in $\not{C} \cap C\ell(K)$. Suppose b is in $\not{C} \cap C\ell(K)$. By Lemma 1.2, we have $\omega\text{-}\lim_n A_n(b) = b$ and $\omega\text{-}\lim_n A_n(b) = 0$; hence $b = 0$.

To see (1.3.2), suppose that $\omega\text{-}\lim_n A_n(b) = b_0$ exists. Then, for each s in S, $(s-I)b_0 = \omega\text{-}\lim_n (s-I)A_n(b) = 0$. Thus $sb_0 - b_0 = 0$, for each s in S; hence b_0 is in $\not{C}$. Consequently, we have

$$\omega\text{-}\lim_n A_n(b-b_0) = \omega\text{-}\lim_n A_n(b) - b_0$$
$$= b_0 - b_0$$
$$= 0,$$

and, by Lemma 1.2, $b - b_0$ is in $C\ell(K)$. Hence $b = (b-b_0)+b_0$ is in $F = \not{c}\oplus C\ell(K)$.

Conversely, suppose that b is in F. By (1.3.1), $b = b_1 + b_2$, for a unique b_1 in $\not{c}$ and a unique b_2 in $C\ell(K)$. It follows from Lemma 1.2 that

$$\omega\text{-}\lim_n A_n(b) = \omega\text{-}\lim_n A_n(b_1) + \omega\text{-}\lim_n A_n(b_2)$$
$$= b_1 + 0$$
$$= b_1.$$

Hence, $\omega\text{-}\lim_n A_n(b)$ exists.

Now, we show that (1.3.3) holds. Since (1.3.1) holds, P is well-defined and linear. Also, for each n, $\|A_n\| \leq \sup \{\|s\|: s \text{ in } S\}$ and hence $\sup_n\|A_n\| \leq \sup\{\|s\|: s \text{ in } S\}$. For each b in F, $\{A_n(b)\}$ is a family of linear functional on B^*. Since, for each β in B^*, $\{\beta[A_n(b)]\}$ is bounded and $\beta[P(b)] = \lim_n \beta[A_n(b)]$, we have $P(b)$ is bounded. (See [7, Theorem 18, p. 55].) Furthermore, $|\beta[P(b)]| \leq \sup_n\|\beta\|\,\|A_n\|\,\|b\|$ and hence $\|P(b)\| \leq \sup_n\|A_n\|\,\|b\|$. It follows that $\|P\| \leq \sup_n\|A_n\| \leq \sup\{\|s\|: s \text{ in } S\}$.

Now, we prove the condition (1.3.4). Since every bounded linear transformation from a Banach space into itself is also ω^*-ω^*-continuous, we have, for each s in S,

$$\begin{aligned}(s-I)P(b) &= (s-I)[\omega\text{-}\lim_n A_n(b)] \\ &= \omega\text{-}\lim_n (s-I)A_n(b) \\ &= 0\end{aligned}$$

and
$$\begin{aligned}P(s-I)(b) &= \omega\text{-}\lim_n A_n(s-I)(b) \\ &= 0.\end{aligned}$$

Hence $Ps = sP = P$. It follows that $s(P(b)) = P(b)$, for each b in F and each s in S. Hence $P(b)$ is in $\mathcal{B}$, for each b in F. By Lemma 1.2, we have $P^2(b) = P(b)$, for each b in F. Thus, $P^2 = P$ and this proves (1.3.4).

To see the condition (1.3.5), notice that it follows from (1.3.1), (1.3.2) and (1.3.4) that P is a projection. By Lemma 1.2 we have $P(C\ell(K)) = \{0\}$.

Now, we prove that (1.3.6) holds. Since, for each n, $\{A_n(b)\} \subseteq \omega\text{-}C\ell C_0O(b) = C\ell C_0O(b)$, $P(b)$ is in $C\ell C_0O(b)$. If b_0 is in $C\ell C_0O(b) \cap \mathcal{B}$, then $b - b_0$ is in $C\ell(K)$. Hence

$$0 = P(b-b_0) = P(b) - P(b_0) = P(b) - b_0$$

and $P(b) = b_0$.

To show the condition (1.3.7), let b be in the closure of F. We claim that $\omega\text{-}\lim_n A_n(b)$ exists and hence, by (1.3.2), b is in F. Then F is closed. To prove this claim, let $\{b_\alpha\}$ be a net in F such that $\lim_\alpha \|b_\alpha - b\| = 0$. Let $a_\alpha = P(b_\alpha)$, for each α. By (1.3.3), we have

$$\|a_\alpha - a_\gamma\| \leq \|P\| \, \|b_\alpha - b_\gamma\|,$$

for any indices α and γ. Since $\{b_\alpha\}$ is Cauchy, so is $\{a_\alpha\}$. It follows from the completeness of B that $\lim_\alpha a_\alpha$ exists. Since $\{a_\alpha\}$ is in $\mathcal{C}$ and $\mathcal{C}$ is closed, $\lim_\alpha a_\alpha = a$, for some a in $\mathcal{C}$. Let ϕ in B^* be arbitrary. For each $\varepsilon > 0$, there is an α such that $\|b_\alpha - b\| < \varepsilon/3M\|\phi\|$ and $\|a_\alpha - a\| < \varepsilon/3M$ where $M = \sup_n \|A_n\|$. Then, there is an $n(\alpha,\varepsilon,\phi)$ such that $\|A_n(b_\alpha) - a_\alpha\| < \varepsilon/3\|\phi\|$, whenever $n > n(\alpha,\varepsilon,\phi)$. Then, whenever $n > n(\alpha,\varepsilon,\phi)$,

$$\begin{aligned} |\phi(A_n(b)-a)| &\leq |\phi(A_n(b-b_\alpha))| + |\phi(A_n(b_\alpha)-a_\alpha)| + |\phi(a_\alpha - a)| \\ &\leq \|\phi\|\,\|A_n\|\,\|b-b_\alpha\| + \|\phi\|\,\|A_n(b_\alpha)-a_\alpha\| \\ &\qquad + \|\phi\|\,\|a_\alpha - a\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

This implies that $\lim_n \phi[A_n(b)-a] = 0$, for any arbitrary ϕ in B^*. Hence $\omega\text{-}\lim_n A_n(b) = a$ and this completes the proof.

1.4. Remark. In the previous theorem, if we replace the weak ergodicity by strong ergodicity, then we will get the corresponding results in the convergence in the norm-topology instead of the weak topology. This theorem is essentially due to Eberlein

[8].

The following theorem is due to Day [5].

1.5. Theorem. *Let* S *be a semigroup. Then the following conditions are equivalent:*

(1.5.1) S *is amenable;*

(1.5.2) *for any bounded representation* Π *and anti-representation* T *of* S *over a Banach space* B*, the operator semigroups* $\{\Pi_s\text{:s in S}\}$ and $\{T_s\text{:s in S}\}$ *are ergodic (weakly, strongly and uniformly);*

(1.5.3) *for the left and right regular representation of* S *over the Banach space* m(S), $\{\ell_s\text{:s in S}\}$ *and* $\{\gamma_s\text{:s in S}\}$ *are ergodic (weakly, strongly and uniformly).*

Proof. Since (1.5.2) is formally stronger than (1.5.3), it only needs to prove that (1.5.1) implies (1.5.2); and (1.5.3) implies (1.5.1).

We first prove that (1.5.1) implies (1.5.2). Suppose that S is amenable. Then, there is a net $\{\phi_n\}$ of finite means on S such that $\lim_n \|1_s\phi_n-\phi_n\| = \lim_n\|\phi_n 1_s-\phi_n\| = 0$. For each ϕ_n, let $\Pi_{\phi_n} = \sum_{s\varepsilon S}\phi_n(s)\Pi_s$ and $T_{\phi_n} = \sum_{s\varepsilon S}\phi_n(s)T_s$. Then $\{\Pi_{\phi_n}\}$ and $\{T_{\phi_n}\}$ are nets of finite averages of $\{\Pi_s\text{:s in S}\}$ and $\{T_s\text{:s in S}\}$, respectively. Then, it follows from Proposition II.3.2 that, for each s in S,

$$\|\Pi_{\phi_n}\Pi_s-\Pi_{\phi_n}\| \le M\|\phi_n 1_s-\phi_n\|$$

and
$$\|\Pi_s\Pi_{\phi_n}-\Pi_{\phi_n}\| \le M\|1_s\phi_n-\phi_n\|,$$

where $M = \sup\{\|\Pi_s\|: s \text{ in } S\}$. Hence, for each s in S,

$\lim_n \|\Pi_{\phi_n}\Pi_s-\Pi_{\phi_n}\| = \lim_n \|\Pi_s\Pi_{\phi_n}-\Pi_{\phi_n}\| = 0$. Similarly, $\lim_n \|T_sT_{\phi_n}-T_{\phi_n}\| = \lim_n \|T_{\phi_n}-T_s-T_{\phi_n}\| = 0$, for each s in S. Thus, $\{\Pi_s: s \text{ in } S\}$ and $\{T_s: s \text{ in } S\}$ are uniformly ergodic. Consequently, they are also strongly and weakly ergodic.

Now, to prove that (1.5.3) implies (1.5.1). Suppose that $\{\gamma_s: s \text{ in } S\}$ and $\{\ell_s: s \text{ in } S\}$ are weakly ergodic. Let K_ℓ and K_γ be the linear span of $\{f-\ell_s f: f \text{ in } m(S)$ and s in $S\}$ and $\{f-\gamma_s f: f \text{ in } m(S)$ and s in $S\}$, respectively. Let $\mathcal{C}_\ell$ and $\mathcal{C}_\gamma$ be the sets of all common fixed points of ℓ_s and γ_s, s in S, respectively. Let $\mathcal{C}$ denote the subspace of all constant functions of $m(S)$. By the ergodicity of $\{\ell_s: s \text{ in } S\}$ and $\{\gamma_s: s \text{ in } S\}$ and by the fact that $\mathcal{C} \subseteq \mathcal{C}_\ell \cap \mathcal{C}_\gamma$, we have $\mathcal{C}\cap K_\ell = \mathcal{C}\cap K_\gamma = \{0\}$. Hence, 1 is not in K_ℓ. Since, $1 = 1 + 0$ is in $1 + K_\ell$, we have $\operatorname{dist}(0,1+K_\ell) = \inf\{\|1+h\|: h \text{ in } K_\ell\} \le 1$. On the other hand, for each h in K_ℓ, $1 + h$ is in $F = \mathcal{C}_\ell \oplus C\ell(K_\ell)$. Since $\|P\| \le \sup\{\|\ell_s\|: s \text{ in } S\} \le 1$, we have $1 = \|P(1+h)\| \le \|P\|\,\|1+h\| \le \|1+h\|$, for each h in K_ℓ. Hence $\operatorname{dist}(0,1+K_\ell) \ge 1$. Combining the two inequalities, we have $\operatorname{dist}(0,1+K_\ell) = 1$ and hence, S is left amenable. (See Theorem

II.3.6.) Similarly, we have $\mathrm{dist}(0, 1+K_{\gamma}) = 1$ and S is right amenable. This proves that S is amenable.

1.6. Remark. (a) Observe that, as a consequence of Theorem 1.5, the weak, strong and uniform ergodicity of S are equivalent.
(b) In the proof of Theorem 1.5, we actually proved that S is uniformly ergodic under a net of finite averages of S. In such a case, we say that S is restrictedly ergodic. If an operator semigroup S is amenable when it is considered as an abstract semigroup, then S is restrictedly ergodic.

The ergodicity of $\{\gamma_s\text{:s in S}\}$ alone does not imply the amenability of S. For example, let S be a semigroup with more than one element such that st = s, for all s and t in S. Since $\gamma_s = I$, for each s in S. $\{\gamma_s\text{:s in S}\}$ is ergodic. However, S is not left amenable. The following proposition gives a special case when this works.

1.7. Theorem. *If* S *is a semigroup such that the set* $\{\ell_s f\text{:s in S and f in m(S)}\}$ *spans* m(S), *then* $\{\gamma_s\text{:s in S}\}$ *is uniformly restrictedly ergodic if and only if* S *is amenable.*

Proof. If S is amenable, then, by Remark 1.6 (b), $\{\gamma_s\text{:s in S}\}$ is uniformly restrictedly ergodic.

Conversely, suppose that $\{\gamma_s\text{:s in S}\}$ is restrictedly ergodic. Then, as a consequence of the proof of (1.5.3) implies

(1.5.1) in Theorem 1.5, S is right amenable. Let $\{\phi_n\}$ be the net of finite mean such that, for each s in S,

$$\lim_n \| \gamma_s\gamma_{\phi_n}-\gamma_{\phi_n} \| = \lim_n \| \gamma_{\phi_n}\gamma_s-\gamma_{\phi_n} \| = 0.$$

By Proposition II.1.6, there is a subnet $\{\phi_k\}$ of $\{\phi_n\}$ converging to some mean μ in the ω^*-topology of $m(S)$. Also note that $\lim_n \| \gamma_s\gamma_{\phi_k}-\gamma_{\phi_k} \| = 0$, for each s in S. Then, for all s and t in S and each f in $m(S)$,

$$\begin{aligned} 0 &= \lim_k |\gamma_s\gamma_{\phi_k}(f)(t)-\gamma_{\phi_k}(f)(t)| \\ &= \lim_k |\gamma_{\phi_k}(f)(ts)-\gamma_{\phi_k}(f)(t)| \\ &= \lim_k |Q\phi_k(\ell_{ts}f)-Q\phi_k(\ell_t f)| \\ &= \lim_k |Q\phi_k(\ell_s\ell_t f-\ell_t f)| \\ &= |\mu(\ell_s\ell_t f-\ell_t f)|. \end{aligned}$$

(Note that the thrid equality follows from Definition II.2.7.) Hence, $\ell_s^*\mu(\ell_t f) = \mu(\ell_t f)$. Since $\{\ell_s f: s \text{ in } S \text{ and } f \text{ in } m(S)\}$ spans $m(S)$, for each f in $m(S)$, $f = \sum_{i=1}^n \ell_{s_i} f_i$, for some s_i in S and f_i in $m(S)$, $1 \leq i \leq n$. Hence, for each s in S,

$$\begin{aligned} \ell_s^*\mu(f) &= \ell_s^*\mu[\textstyle\sum_{i=1}^n \ell_{s_i} f_i] \\ &= \textstyle\sum_{i=1}^n \ell_s^*\mu(\ell_{s_i} f_i) \\ &= \textstyle\sum_{i=1}^n \mu(\ell_{s_i} f_i) \\ &= \mu(f). \end{aligned}$$

Thus, S is left amenable.

1.8. Consequences. As observed in Remark 1.4 if we replace weak ergodicity by strong ergodicity in Theorem 1.3, then the corresponding results hold for convergence in the norm topology. Furthermore, if there is a weak cluster point b_0 of the net $\{A_n(b)\}$, then, there is a subnet $\{A_k\}$ of $\{A_n\}$ such that $\omega\text{-}\lim_k A_k(b) = b_0$ and

$$\begin{aligned} s(b_0) &= s(\omega\text{-}\lim_k A_k(b)) \\ &= \omega\text{-}\lim_k A_k(b) \\ &= b_0, \end{aligned}$$

for each s in S. (Note that the second equality holds, since the strong ergodicity implies weak ergodicity.) Hence, b_0 is in $C\ell C_0 O(b) \cap F$. Consequently, $\{A_n(b)\}$ converges strongly to b_0. Thus, with Theorem 1.5 and above remark, theorems of von Neumann, Riesz, Yosida and Kakutani follow.

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