Characterizations for the Existence of a Solution to the Moment Problem on a Finite Number of Intervals

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Contents

Abstract

Let E be a subset of real numbers defined by $E = (\cup_{1 \le i \le m}[\alpha_{2i-1}, \alpha_{2i}]) \cap (-\infty, \infty)$, where $-\infty \le \alpha_1 < \alpha_2 < \ldots < \alpha_{2m} \le \infty$. The *moment problem on E* can be stated as: given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, find a distribution function ψ with an infinite spectrum contained in E such that

$$(*) \qquad \int_E x^n \, d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \ldots .$$

A sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ satisfying (*) will be called a *moment sequence on E*. That is, $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is called a moment sequence on E if there exists a distribution function ψ such that (*) is satisfied.

If E is a finite interval $[\alpha_1, \alpha_2]$, then we have essentially the Hausdorff moment problem and the corresponding sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ satisfying (*) will be called a *Hausdorff moment sequence* on $[\alpha_1, \alpha_2]$. When E is a union of two semi-infinite disjoint intervals, say $(-\infty, \alpha_2] \cup [\alpha_3, \infty)$, we have the complemented Hausdorff moment problem and we called the corresponding moment sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ a *complemented Hausdorff moment sequence* on $(-\infty, \alpha_2] \cup [\alpha_3, \infty)$.

Let us define the determinants $\Delta_n\{\sum_{i=0}^{p} a_i x^i\}$ by

$$\Delta_n\{\sum_{i=0}^{p} a_i x^i\} = \begin{vmatrix} \sum_{i=0}^{p} a_i\mu_i & \cdots & \sum_{i=0}^{p} a_i\mu_{i+n} \\ \cdots & \cdots & \cdots \\ \sum_{i=0}^{p} a_i\mu_{i+n} & \cdots & \sum_{i=0}^{p} a_i\mu_{i+2n} \end{vmatrix} .$$

The main purpose of this thesis is to find characterizations for the existence of a solution to the moment problem on various sets E. That is, to find characterizations for moment sequences on various sets E. This utilizes the Δ_n's.

In Chapter 1, we introduce the moment problem and the general theory of orthogonal polynomials. At the end of Chapter 1, we use this general theory of orthogonal polynomials to prove the well known characterization, $\Delta_n\{1\} > 0$, for the existence of a solution to the Hamburger moment problem (i.e. $E = (-\infty, \infty)$).

In Chapter 2 we give (i) some representation theorems for polynomials that are non-negative on the set E and (ii) additional representation theorems for the moment functional associated with various moment problems. These polynomial representation theorems and moment functional representation theorems are used in the later chapters to find characterizations for moment sequences on various sets E.

Chapter 3 discusses the Stieltjes ($E = [0, \infty)$) and the Hausdorff ($E = [0, 1]$) moment problems.

The main new results of the thesis are in Chapter 4 and Chapter 5.

In Chapter 4 we give two new characterizations for a Hausdorff moment sequence and a characterization for a complemented Hausdorff moment sequence. These results can be stated as follows:

(I) $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a Hausdorff moment sequence on $[\alpha_1, \alpha_2]$ if and only if

$$\Delta_n\{1\} > 0,\ \Delta_n\{x - \alpha_1\} > 0,\ \Delta_n\{\alpha_2 - x\} > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots ;$$

if and only if

$$\Delta_n\{1\} > 0,\ \Delta_n\{(x - \alpha_1)(\alpha_2 - x)\} > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

(II) $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a complemented Hausdorff moment sequence on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ if and only if

$$\Delta_n\{1\} > 0,\ \Delta_n\{(x - \alpha_2)(x - \alpha_3)\} > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

In Chapter 5 we give some characterizations for the moment sequence on E where E is a finite union of intervals. They can be stated as follows:

(I) Let $-\infty < \alpha_1 < \alpha_{2m} < \infty$. Then $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0, \Delta_n\{(x - \alpha_1)(\alpha_{2m} - x)\} > 0,$$

$$\Delta_n\{\prod_{i=2}^{2m-1}(x - \alpha_i)\} > 0 \quad \text{and} \quad \Delta_n\{-\prod_{i=1}^{2m}(x - \alpha_i)\} > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

(II) Let $\mathbf{A} = \{A_s(x) \mid A_s(x) = d_s\Pi_{i\in S}(x - \alpha_i), |\alpha_i| < \infty, S \subseteq \{1, ..., 2m\}, d_s = \pm 1,$ $A_s(x) \geq 0$ for $x\in E\}$. $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{A_s(x)\} > 0,\ n = 0, 1, 2, \ldots ,\ \text{for all}\ A_s(x)\in \mathbf{A}.$$

(III) Let $\{\mu_n \mid n = 0, 1, 2, ...\}$ be a moment sequence which is associated with a determinate moment problem on E, where $-\infty < \alpha_1$ and $\alpha_{2m} = \infty$. $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0,\ \Delta_n\{x - \alpha_1\} > 0 \text{ and } \Delta_n\{(x - \alpha_{2i})(x - \alpha_{2i+1})\} > 0 \quad i = 1, \ldots, m-1.$$

Chapter 1

Moment Problem and Orthogonal Polynomials

1.1 Introduction

Let **R** be the set of real numbers. A bounded non-decreasing function ψ: **R** $\to$ **R** is called a *distribution function* if its *moments* μ_n, defined by the Riemann-Stieltjes integral

$$\mu_n = \int_{-\infty}^{\infty} x^n \, d\psi(x), \quad n = 0, 1, 2, \dots ,$$

are all finite. Without loss of generality we require that the distribution functions be continuous from the right at each point of **R**. The set $\sigma(\psi)$ defined by

$$\sigma(\psi) = \{ x \mid \psi(x + \delta) - \psi(x - \delta) > 0 \text{ for all } \delta > 0 \}$$

is called the *spectrum* of ψ. A point in $\sigma(\psi)$ is called a *spectral point* of ψ.

Let E be a subset of the real numbers. The moment problem on E can be stated in the following manner: given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \dots\}$, find a distribution function ψ with an infinite spectrum contained in E such that

$$\int_E x^n \, d\psi(x) = \mu_n , \quad n = 0,1,2, \dots .$$

There are three questions associated with any moment problem. They are: (i) to construct all distribution functions that are solutions of the moment problem, (ii) to give necessary and sufficient conditions for the existence of a solution of the moment problem, and (iii) to give necessary and sufficient conditions for the uniqueness of the solution of the moment problem. In this thesis we only deal with the question of existence of a solution for the moment problem on various sets E.

The moment problem had its beginnings back in 1874 with the investigations of P. L. Tchebichef [1] and his pupil A. Markov [1].

In 1894-95, T. J. Stieltjes [1] proposed the moment problem on $[0,\infty)$. That is, given a real sequence $\{\mu_n \mid n = 0, 1, 2, ...\}$, find a distribution function ψ with an infinite spectrum contained in $[0, \infty)$ such that

$$\int_0^\infty x^n d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \dots \; . \qquad (1.1.1)$$

This moment problem is now called the *Stieltjes moment problem* .

In order to pose the problem in this generality, Stieltjes had to invent the Stieltjes integral and introduced many new and important ideas into analysis.

In 1920-21, H. Hamburger [1] made an important extension of the problem by allowing the spectrum of ψ to be in $(-\infty, \infty)$. That is, he posed the problem of finding a distribution function ψ such that

$$\int_{-\infty}^\infty x^n d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \dots \; . \qquad (1.1.2)$$

This problem is now known as the *Hamburger moment problem* .

In 1923, F. Hausdorff [1] investigated the moment problem on $[0,1]$, that is to find a distribution function ψ with an infinite spectrum contained in $[0,1]$, such that

$$\int_0^1 x^n \, d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \dots . \qquad (1.1.3)$$

This problem is now referred to as the *Hausdorff moment problem* .

As can be seen from the works of Stieltjes, Hamburger and Hausdorff the moment problem and the theory of orthogonal polynomials are closely related. We cannot discuss the moment problem without discussing the theory of orthogonal polynomials. In the next several sections, we will provide the necessary materials about orthogonal polynomials which will be used for our investigation of the moment problem.

In order that the thesis is self-contained we have included the next two sections which deal with orthogonal polynomials and section 1.4 and 1.5 that deal with some related questions in analysis. Unless otherwise stated all the definitions, lemmas and theorems in sections 1.2 to 1.5 can be found in Chihara's text [1].

1.2 Some Properties of Orthogonal Polynomials

These results on orthogonal polynomials will be used at the end of this chapter and in the later chapters to characterize the existence of a solution to the moment problem whose spectrum is contained in various sets E.

We start with a discussion of the linear functional L and the corresponding orthogonal polynomials.

Definition 1.2.1 Let $\{\mu_n \mid n = 0, 1, 2, \dots\}$ be a sequence of real numbers and let L be a linear functional defined on the vector space of all polynomials by

$$L[x^n] = \mu_n, \quad n = 0,1\ 2, \dots$$

$$L\,[\alpha_1\pi_1(x) + \alpha_2\pi_2(x)] = \alpha_1 L\,[\pi_1(x)] + \alpha_2 L\,[\pi_2(x)]\,,$$

for all real numbers α_i and all polynomials $\pi_i(x)$ (i=1,2). Then L is called the *moment functional* determined by the sequence $\{\mu_n \mid n = 0, 1, 2, ...\}$. The number μ_n is called the moment of order n of L.

Definition 1.2.2 A sequence $\{P_n(x)\}$ is called an *orthogonal polynomial sequence* with respect to a moment functional L, if for all non-negative integers m and n,

(i) $P_n(x)$ is a polynomial of degree n,

(ii) $L\,[P_m(x)P_n(x)] = 0$ for $m \neq n$,

(iii) $L\,[P_n^2(x)] \neq 0$.

"OPS" will be the abreviation used for "Orthogonal polynomial sequence" and we will use the phrase "$\{P_n(x)\}$ is an OPS for L" for any polynomial sequence $\{P_n(x)\}$ that satisfies Definition 1.2.2.

If $\{P_n(x)\}$ is an OPS for L and in addition we also have the leading coefficient of $P_n(x)$ equal to one for all n, then $\{P_n(x)\}$ will be called a *monic OPS* for L.

Let δ_{mn} be the Kronecker delta defined by

$$\delta_{mn} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n\,. \end{cases} \qquad (1.2.1)$$

Conditions (i) and (ii) of Definition 1.2.2 can be replaced by

$$L[P_m(x)P_n(x)] = K_n\delta_{mn}, \qquad K_n \neq 0. \qquad (1.2.2)$$

Theorem 1.2.3 Let L be a moment functional and let $\{P_n(x)\}$ be a sequence of polynomials. Then the following are equivalent:

(a) $\{P_n(x)\}$ is an OPS for L;

(b) $L[\pi(x)P_n(x)] = 0$ for every polynomial $\pi(x)$ of degree $m < n$,

while $L[\pi(x)P_n(x)] \neq 0$ if $m = n$;

(c) $L[x^m P_n(x)] = K_n \delta_{mn}$ where $K_n \neq 0$, $m = 0, 1, ..., n$; $n = 0, 1, 2, ...$.

Proof: Let $\{P_n(x)\}$ be an OPS for L. Since each $P_k(x)$ is of degree k, it is clear that $\{P_0(x), P_1(x), ..., P_m(x)\}$ is a basis for the vector subspace of polynomials of degree at most m. Thus if $\pi(x)$ is a polynomial of degree m, there exist constants c_k such that

$$\pi(x) = \sum_{k=0}^{m} c_k P_k(x), \qquad c_m \neq 0 .$$

By the linearity of L,

$$L[\pi(x) P_n(x)] = \begin{cases} \sum_{k=0}^{m} c_k L[P_k(x) P_n(x)] = 0 & \text{if } m < n \\ c_n L[P_n^2(x)] & \text{if } m = n. \end{cases}$$

Thus (a) $\Rightarrow$ (b) . Since trivially (b) $\Rightarrow$ (c) $\Rightarrow$ (a), thus the proof is complete.

Q.E.D.

Definition 1.2.4 Let $E \subseteq (-\infty, \infty)$. A moment functional L is said to be *positive-definite on E* if and only if $L[\pi(x)] > 0$ for every real polynomial $\pi(x)$ which is non-negative on E and does not vanish identically.

If in this definition $E = (-\infty, \infty)$, then L is said to be *positive-definite.* Also we say that L is *non-negative-definite on E* if in Definition 1.2.4

$L[\pi(x)] > 0$ is replaced by $L[\pi(x)] \geq 0$.

Lemma 1.2.5 Let $\pi(x)$ be a polynomial that is non-negative for all real x. Then there are real polynomials p(x) and q(x) such that

$$\pi(x) = p^2(x) + q^2(x).$$

Proof: If $\pi(x) \geq 0$ for real x, then $\pi(x)$ is a real polynomial so its real zeros have even multiplicity and its non-real zeros occur in conjugate pairs. Thus we can write

$$\pi(x) = r^2(x) \prod_{k=1}^{m} (x - \alpha_k - \beta_k i)(x - \alpha_k + \beta_k i) ,$$

where r(x) is a polynomial, α_k and β_k are real numbers.

Writing

$$\prod_{k=1}^{m} (x - \alpha_k - \beta_k i) = A(x) + i\, B(x)$$

where A(x) and B(x) are real polynomials, we get

$$\pi(x) = r^2(x) [A^2(x) + B^2(x)].$$

Q.E.D.

In order to discuss existence theorem for OPS, we introduce the determinants

$$\Delta_n\{\sum_{i=0}^{p} a_i x^i\} = \begin{vmatrix} \sum_{i=0}^{p} a_i\mu_i & \cdots & \sum_{i=0}^{p} a_i\mu_{i+n} \\ \cdots & \cdots & \cdots \\ \sum_{i=0}^{p} a_i\mu_{i+n} & \cdots & \sum_{i=0}^{p} a_i\mu_{i+2n} \end{vmatrix} . \quad (1.2.3)$$

Definition 1.2.6 The real quadratic form

$$\sum_{i,j=0}^{n} a_i a_j \mu_{i+j}$$

is called positive-definite. If for any vector $(a_0, ..., a_n) \neq (0, ..., 0)$, we have

$$\sum_{i,j=0}^{n} a_i a_j \mu_{i+j} > 0 .$$

The following well known result is from linear algebra.

Lemma 1.2.7 (Archbold [1] P.393) The real quadratic form

$$\sum_{i,j=0}^{n} a_i a_j \mu_{i+j}$$

is positive-definite if and only if $\Delta_m\{1\} > 0$ for $m = 0, ..., n$.

Using this lemma we have the following theorem.

Theorem 1.2.8 Let $\{\mu_n \mid n = 0, 1, 2, ...\}$ be a real sequence and let the linear functional L be defined by $L[x^n] = \mu_n$. Then L is positive-definite if and only if $\Delta_n\{1\} > 0$, $n = 0, 1, 2, ...$.

Proof: Suppose that L is positive-definite. Given $n \geq 0$, and any vector $(a_0, ..., a_n) \neq (0, ..., 0)$, and

$$\pi(x) = \sum_{i=0}^{n} a_i x^i ,$$

we have that

$$0 < L[\pi^2(x)] = L[(\sum_{i=0}^{n} a_i x^i)^2] = L[\sum_{i,j=0}^{n} a_i a_j x^{i+j}] = \sum_{i,j=0}^{n} a_i a_j \mu_{i+j} .$$

By Lemma 1.2.7, we have $\Delta_n\{1\} > 0$.

Conversely, let $\Delta_n\{1\} > 0$ for $n = 0, 1, 2, ...$, and let $\pi(x)$ be a polynomial that is non-negative and does not vanish identically on **R**. By Lemma 1.2.5 we know that

$$\pi(x) = p^2(x) + q^2(x) = (\sum_{i=0}^{m} a_i x^i)^2 + (\sum_{k=0}^{n} b_k x^k)^2.$$

Therefore,

$$L[\pi(x)] = L[(\sum_{i=0}^{m} a_i x^i)^2] + L[(\sum_{k=0}^{n} b_k x^k)^2] = L[\sum_{i,j=0}^{m} a_i a_j x^{i+j}] + L[\sum_{k,t=0}^{n} b_k b_t x^{k+t}]$$

$$= \sum_{i,j=0}^{m} a_i a_j \mu_{i+j} + \sum_{k,t=0}^{n} b_k b_t \mu_{k+t}.$$

By Lemma 1.2.7, we have $L[\pi(x)] > 0$.

Q.E.D.

Theorem 1.2.9 Let L be a positive-definite moment functional with real moment sequence $\{\mu_n \mid n = 0, 1, 2, ...\}$. Then there exist an OPS for L.

Proof: Write

$$P_n(x) = \sum_{k=0}^{n} c_{nk} x^k.$$

Recalling Theorem 1.2.3, we observe that the orthogonality conditions

$$L[x^m P_n(x)] = \sum_{k=0}^{n} c_{nk}\mu_{k+m} = K_n \delta_{mn}, \quad K_n \neq 0, \quad m \le n, \quad (1.2.4)$$

are equivalent to the matrix equation

$$\begin{bmatrix} \mu_0 \ \mu_1 \cdots \mu_n \\ \mu_1 \ \mu_2 \cdots \mu_{n+1} \\ \cdots \ \cdots \ \cdots \ \cdots \\ \mu_n \ \mu_{n+1} \cdots \mu_{2n} \end{bmatrix} \begin{bmatrix} c_{n0} \\ c_{n1} \\ \cdots \\ c_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ K_n \end{bmatrix} \qquad (1.2.5)$$

Since L is positive-definite, by Theorem 1.2.8 we have $\Delta_n\{1\} > 0$. So for arbitrary $K_n \neq 0$, (1.2.5) has a unique solution $\{c_{ni} \mid i = 0, ..., n\}$. Thus there exists $P_n(x)$ satisfying (1.2.4). We also have

$$c_{nn} = \frac{K_n \Delta_{n-1}\{1\}}{\Delta_n\{1\}} \neq 0, \qquad n \ge 1, \qquad (1.2.6)$$

which is valid for $n = 0$, if we define $\Delta_{-1}\{1\} = 1$. It follows that $P_n(x)$ is of degree n, hence $\{P_n(x)\}$ is an OPS for L.

Q.E.D.

Theorem 1.2.10 Let $\{P_n(x)\}$ be an OPS for L. Then for any polynomial $\pi_n(x)$ of degree n,

$$L[\pi_n(x)\, P_n(x)\,] = a_n L\,[\, x^n P_n(x)\,] = \frac{a_n k_n \Delta_n\{1\}}{\Delta_{n-1}\{1\}}, \quad \Delta_{-1}\{1\} = 1, \quad (1.2.7)$$

where a_n is the leading coefficient of $\pi_n(x)$ and k_n is the leading coefficient of $P_n(x)$.

Proof: Writing

$$\pi_n(x) = a_n x^n + \pi_{n-1}(x)\ ,$$

where $\pi_{n-1}(x)$ is a polynomial of degree n - 1, we have

$$L[\pi_n(x)P_n(x)] = a_n\, L[x^n P_n(x)] + L[\pi_{n-1}(x)P_n(x)] = a_n L[x^n P_n(x)].$$

Thus (1.2.7) follows from (1.2.6) with $k_n = c_{nn}$.

Q.E.D.

One of the most important characteristics of orthogonal polynomials is the fact that any three consecutive orthogonal polynomials are connected by a very simple relation which we now derive.

Theorem 1.2.11 Let L be a positive-definite moment functional and let $\{P_n(x)\}$ be the corresponding monic OPS. Then there exist constants c_n and $\lambda_n > 0$ such that

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n = 1, 2, 3, \dots, \quad (1.2.8)$$

where we define $P_{-1}(x) = 0$.

Proof: Since $xP_n(x)$ is a polynomial of degree n +1, we can write

$$xP_n(x) = \sum_{k=0}^{n+1} a_{nk}P_k(x), \qquad a_{nk} = \frac{L[xP_n(x)P_k(x)]}{L[P_k^2(x)]}.$$

But $xP_k(x)$ is a polynomial of degree k +1 so that $a_{nk} = 0$ for $0 \le k < n - 1$.

Further, $xP_n(x)$ is monic so $a_{n,n+1} = 1$. Thus

$$xP_n(x) = P_{n+1}(x) + a_{nn}P_n(x) + a_{n,n-1}P_{n-1}(x), \qquad n \ge 1.$$

By replacing n by n - 1 in this equation we obtain

$$xP_{n-1}(x) = P_n(x) + c_nP_{n-1}(x) + \lambda_nP_{n-2}(x), \qquad n \ge 2.$$

and this is equivalent to (1.2.8) for $n \ge 2$. If we define $P_{-1}(x) = 0$ and choose $c_1 = - P_1(0)$, then (1.2.8) is valid also for n = 1.

Next from (1.2.8) we obtain

$$L[x^{n-2}P_n(x)] = L[x^{n-1}P_{n-1}(x)] - c_nL[x^{n-2}P_{n-1}(x)] - \lambda_nL[x^{n-2}P_{n-2}(x)],$$

$$0 = L[x^{n-1}P_{n-1}(x)] - \lambda_nL[x^{n-2}P_{n-2}(x)].$$

By Theorem 1.2.10, we obtain for $n \ge 1$,

$$\lambda_{n+1} = \frac{L[x^nP_n(x)]}{L[x^{n-1}P_{n-1}(x)]} = \frac{\Delta_{n-2}\{1\}\Delta_n\{1\}}{\Delta_{n-1}^2\{1\}} \qquad (\Delta_{-1}\{1\} = 1).$$

By Theorem 1.2.8 we know that L is positive-definite implies $\Delta_n\{1\} > 0$.

Therefore $\lambda_n > 0$ for n = 1, 2, ...

Q.E.D.

Let $P'_n(x)$ be the derivative of $P_n(x)$ with respect to x. We have the following theorem.

Theorem 1.2.12 Let L be a positive-definite moment functional and $\{P_n(x)\}$ be an OPS for L. Then

$$\sum_{k=0}^{n} \frac{P_k^2(x)}{\lambda_1\lambda_2 \dots \lambda_{k+1}} = \frac{P'_{n+1}(x)\, P_n(x) \;-\; P'_n(x)\, P_{n+1}(x)}{\lambda_1\lambda_2 \dots \lambda_{n+1}} . \qquad (1.2.9)$$

Proof: From (1.2.8), we have for $n \geq 0$ the identities

$$xP_n(x)P_n(u) = P_{n+1}(x)P_n(u) + c_{n+1}P_n(x)P_n(u) + \lambda_{n+1}P_{n-1}(x)P_n(u),$$

$$uP_n(u)P_n(x) = P_{n+1}(u)P_n(x) + c_{n+1}P_n(u)P_n(x) + \lambda_{n+1}P_{n-1}(u)P_n(x).$$

Subtracting the second equation from the first yields

$$(x - u)P_n(x)P_n(u) = P_{n+1}(x)P_n(u) - P_{n+1}(u)P_n(x)$$

$$- \lambda_{n+1}[P_n(x)P_{n-1}(u) - P_n(u)P_{n-1}(x)].$$

Let

$$F_n(x,u) = (\lambda_1 \dots \lambda_{n+1})^{-1} \frac{P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u)}{x - u} ,$$

then the last equation can be rewritten

$$\frac{P_m(x)\, P_m(u)}{\lambda_1\lambda_2 \dots \lambda_{m+1}} = F_m(x,u) - F_{m-1}(x,u) , \qquad m \geq 0 .$$

Summing the above in m from 0 to n and noticing that $F_{-1}(x,u) = 0$, we obtain

$$\sum_{k=0}^{n} \frac{P_k(x)P_k(u)}{\lambda_1\lambda_2 \dots \lambda_{k+1}} = (\lambda_1\lambda_2 \dots \lambda_{n+1})^{-1} \frac{P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u)}{x - u} . \qquad (1.2.10)$$

The numerator of the right side of (1.2.10) can be written

$P_{n+1}(x)P_n(u) - P_n(x)P_{n+1}(u) = [P_{n+1}(x) - P_{n+1}(u)]P_n(x) - [P_n(x) - P_n(u)]P_{n+1}(x)$ so (1.2.9) follows from (1.2.10) by letting $u \to x$.

Q.E.D.

As an immediate corollary, we obtian the important inequality

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0, \qquad (1.2.11)$$

valid for all real x whenever L is positive-definite.

1.3 Zeros of OPS and Gauss Quadrature

When the moment functional is positive-definite, the zeros of the corresponding orthogonal polynomials exhibit a certain regularity in their behavior.

Theorem 1.3.1 If L is positive-definite on E and E is an infinite set, then L is positive-definite on every set containing E.

Proof: Let $\pi(x)$ be a real polynomial which is non-negative on the set S and does not vanish identically. If $E \subset S$, then trivially, $\pi(x) \geq 0$ on E. It follows that $L[\pi(x)] > 0$.

Q.E.D.

Theorem 1.3.2 Let I be an interval and L be positive-definite on I. The zeros of $P_n(x)$ are all real, simple and are located in the interior of I.

Proof: Since $L[P_n(x)] = 0$, $P_n(x)$ must change sign at least once in the interval I. That is, $P_n(x)$ has at least one zero of odd multiplicity located in the interior of I.

Let $x_1, x_2, \ldots, x_k$ denote the distinct zeros of odd multiplicity that are located in the interior of I. Set

$$\rho(x) = (x - x_1) \ldots (x - x_k).$$

Then $\rho(x)P_n(x)$ is a polynomial that has no zeros of odd multiplicity in the interior of I, hence $\rho(x)P_n(x) \geq 0$ for $x \in$ I. Therefore, $L[\rho(x)P_n(x)] > 0$. But this contradicts Theorem 1.2.3 unless $k \geq n$. That is, $k = n$ so $P_n(x)$ has n distinct zeros in the interior of I.

Q.E.D.

Henceforth, we denote the zeros of $P_n(x)$ by x_{ni}, the zeros being ordered by increasing magnitude:

$$x_{n1} < x_{n2} < \dots < x_{nn}, \qquad n \geq 1. \qquad (1.3.1)$$

Without loss of generality we assume that $P_n(x)$ has positive leading coefficient, it then follows that

$$P_n(x) > 0 \quad \text{for } x > x_{nn}; \quad \operatorname{sgn} P_n(x) = (-1)^n \quad \text{for } x < x_{n1.} \quad (1.3.2)$$

Here sgn denotes the signum function defined by

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} .$$

Now the derivative of $P_n(x)$, $P'_n(x)$, has at least one, hence exactly one, zero in each of the intervals, $(x_{n,k-1}, x_{nk})$. It follows that $P'_n(x_{nk})$ alternates in sign as k goes from 1 to n. Since $P'_n(x)$ also has positive leading coefficient, we can conclude:

$$\operatorname{sgn} P'_n(x_{nk}) = (-1)^{n-k}, \qquad k = 1, 2, \dots, n. \qquad (1.3.3)$$

Theorem 1.3.3 (Separation theorem for the zeros) The zeros of $P_n(x)$ and $P_{n+1}(x)$ mutually separate each other. That is,

$$x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \qquad i = 1, 2, \dots, n. \qquad (1.3.4)$$

Proof: We have the inequality (1.2.11)

$$P'_{n+1}(x)P_n(x) - P'_n(x)P_{n+1}(x) > 0$$

In particular,

$$P'_{n+1}(x_{n+1,k})P_n(x_{n+1,k}) > 0, \qquad k = 1, 2, \dots, n+1. \qquad (1.3.5)$$

Referring to (1.3.3), we conclude that $\operatorname{sgn} P_n(x_{n+1,k}) = (-1)^{n+1-k}$. Thus $P_n(x)$ has at least one, hence exactly one, zero in each of the n intervals, $(x_{n+1,k}, x_{n+1,k+1})$. (k = 1, 2, ..., n).

Q.E.D.

Corollary 1.3.4 For each $k \geq 1$, $\{x_{nk} \mid n = k, k+1, \ldots\}$ is a decreasing sequence in n and $\{x_{n,n-k+1} \mid n = k, k+1, \ldots\}$ is an increasing sequence in n. In particular, the limits

$$\xi_i = \lim_{n\to\infty} x_{ni}, \quad \eta_j = \lim_{n\to\infty} x_{n,n-j+1}, \quad i,j = 1, 2, 3, \ldots , \qquad (1.3.6)$$

all exist in the extended real number system $[-\infty, \infty]$.

Definition 1.3.5 The closed interval, $[\xi_1, \eta_1]$ is called the *true interval of orthogonality* of the OPS $\{P_n(x)\}$.

Theorem 1.3.6 (Gauss quadrature formula). Let L be positive-definite. There are numbers $A_{n1}, A_{n2}, \ldots , A_{nn}$ such that for every polynomial $\pi(x)$ of degree at most $2n - 1$,

$$L[\pi(x)] = \sum_{k=1}^{n} A_{nk}\pi(x_{nk}) . \qquad (1.3.7)$$

The numbers A_{nk} are all positive and satisfy the condition,

$$A_{n1} + A_{n2} + \ldots + A_{nn} = \mu_0 . \qquad (1.3.8)$$

Proof: Let $\pi(x)$ be an arbitrary polynomial whose degree does not exceed $2n - 1$, and construct the Lagrange interpolation polynomial $L_n(x)$ which corresponds to the nodes x_{nk} and the ordinates $\pi(x_{nk})$ $(1 \leq k \leq n)$. Thus consider

$$L_n(x) = \sum_{k=1}^{n} \pi(x_{nk})\, f_k(x)$$

where

$$f_k(x) = \frac{P_n(x)}{(x - x_{nk})P'_n(x_{nk})} .$$

Now $Q(x) = \pi(x) - L_n(x)$ is a polynomial of degree at most $2n - 1$ which vanishes at x_{nk} (k = 1, ... , n). That is,

$$Q(x) = R(x)P_n(x)$$

where R(x) is a polynomial of degree at most n - 1. By Theorem 1.2.3,

$$L[\pi(x)] = L[L_n(x)] + L[R(x) P_n(x)] = L[L_n(x)] = \sum_{k=1}^{n} \pi(x_{nk}) L[f_k(x)] .$$

This yields (1.3.7) with $A_{nk} = L[f_k(x)]$.

If the particular choice $\pi(x) = f_m^2(x)$ is made in (1.3.7), the result is

$$0 < L[f_m^2(x)] = \sum_{k=1}^{n} A_{nk} f_m^2(x_{nk}) = A_{nm}$$

so the A_{nm} are all positive.

Finally, (1.3.8) can be obtained by choosing $\pi(x) = 1$ in (1.3.7).

Q.E.D.

1.4 Helly's Theorems

We first establish some general convergence theorems which will be used in the proof of the representation theorem (See Theorem 1.5.2).

Theorem 1.4.1 Let $\{f_n\}$ be a sequence of real functions defined on a countable set E. If for each $x \in E$, $\{f_n(x)\}$ is bounded, then $\{f_n\}$ contains a subsequence that converges everywhere on E.

Proof: Let $E = \{ x_1, x_2, x_3, \ldots \}$ and write $f_n^{(0)} = f_n$. Now since $\{f_n^{(0)}(x_1)\}$ is a bounded sequence of real numbers, it contains a convergent subsequence. That is, there is a sequence $\{f_n^{(1)}\}$ which converges for $x = x_1$ and is a subsequence of $\{f_n^{(0)}\}$.

Now $\{f_n^{(1)}(x_2)\}$ is a bounded sequence so we can conclude as before that there is a subsequence $\{f_n^{(2)}\}$ of $\{f_n^{(1)}\}$ which converges for $x = x_2$. Continuing in this way, we obtain sequences $\{f_n^{(0)}\}$ $\{f_n^{(1)}\}$... $\{f_n^{(i)}\}$, ... such that

(a) $\{f_n^{(k)}\}$ is a subsequence of $\{f_n^{(k-1)}\}$ ($k = 1, 2, 3, \ldots$);

(b) $\{f_n^{(k)}(x)\}$ converges for $x \in E_k = \{ x_1, \ldots , x_k \}$.

It follows from (a) that the diagonal sequence , $\{f_n^{(n)}\}$, is a subsequence of $\{f_n\}$. Since, except possibly for the first $k - 1$ terms, $\{f_n^{(n)}\}$ is also a subsequence of $\{f_n^{(k)}\}$, it follows from (b) that $\{f_n^{(n)}(x)\}$ converges for $x \in E = \cup_{1 \leq k < \infty} E_k$.

Q.E.D.

We next prove a theorem which, when stated in terms of functions of bounded variation, is usually known as Helly's Selection Principle (or Theorem of Choice). For the problems considered in this thesis we need it only for the case of non-decreasing functions.

Theorem 1.4.2 Let $\{\phi_n\}$ be a uniformly bounded sequence of non-decreasing functions definied on $(-\infty, \infty)$. Then $\{\phi_n\}$ has a subsequence which converges pointwise on $(-\infty, \infty)$ to a bounded, non-decreasing function.

Proof: Let Q denote the rational numbers. According to Theorem 1.4.1 there is a subsequence $\{\phi_{n(k)}\}$ which converges everywhere on Q. We then define a function ϕ^* on Q by

$$\phi^*(r) = \lim_{k \to \infty} \phi_{n(k)}(r) \qquad \text{for} \quad r \in Q .$$

It follows from the conditions on $\{\phi_n\}$ that ϕ^* is bounded and non-decreasing on Q. We now extend the domain of ϕ^* to **R** by defining

$$\phi^*(x) = \sup \{ \phi^*(r) \mid r \in Q \text{ and } r < x \}, \qquad x \in \mathbf{R} \setminus Q .$$

ϕ^* is clearly bounded and non-decreasing on **R**. We next show that $\{\phi_{n(k)}\}$ converges to $\phi^*(x)$ at all points x of continuity of ϕ^*.

To this end, suppose ϕ^* is continuous at $x \notin Q$. Since Q is dense in **R**, then given $\varepsilon > 0$, there is an $x_2 > x$, $x_2 \in Q$, such that

$$\phi^*(x_2) \leq \phi^*(x) + \varepsilon .$$

For any $x_1 \in Q$, $x_1 < x$, we also have

$$\phi_{n(k)}(x_1) \leq \phi_{n(k)}(x) \leq \phi_{n(k)}(x_2) .$$

Thus

$$\phi^*(x_1) \leq \liminf_{k\to\infty} \phi_{n(k)}(x) \leq \limsup_{k\to\infty} \phi_{n(k)}(x) \leq \phi^*(x_2) .$$

Therefore

$$\phi^*(x) \leq \liminf_{k\to\infty} \phi_{n(k)}(x) \leq \limsup_{k\to\infty} \phi_{n(k)}(x) \leq \phi^*(x) + \varepsilon ,$$

whence it follows that $\{\phi_{n(k)}\}$ converges to ϕ^* at all points of continuity of ϕ^*. But ϕ^* is non-decreasing so its points of discontinuity form a countable set D. Applying Theorem 1.4.1 to $\{\phi_{n(k)}\}$ and D, we conclude that there is a subsequence of $\{\phi_{n(k)}\}$ which converges on D, hence on **R**, to a limit function ϕ (ϕ is of course identical with ϕ^* on $\mathbf{R}\setminus D$). The conditions on $\{\phi_n\}$ guarantee that ϕ is bounded and non-decreasing.

Q.E.D.

Next, we prove Helly's second theorem. As before, we consider only non-decreasing functions.

Theorem 1.4.3 Let $\{\phi_n\}$ be a uniformly bounded sequence of non-decreasing functions defined on a compact interval [a, b], and let it converge pointwise on [a, b] to a limit function ϕ. Then for every real function f continuous on [a, b],

$$\lim_{n\to\infty}\int_a^b f\, d\phi_n = \int_a^b f\, d\phi .$$

Proof: Since $\{\phi_n\}$ is uniformly bounded, there exists an M > 0 such that

$$0 \le \phi_n(b) - \phi_n(a) \le M, \qquad n = 1, 2, 3, \ldots ,$$

hence also

$$0 \le \phi(b) - \phi(a) \le M.$$

Let $\varepsilon > 0$ be given. If f is real and continuous on [a, b], then f is uniformly continuous on [a, b] so there is a partition $P_\varepsilon = \{ x_0, x_1, \ldots, x_m \}$ of [a, b] such that

$$| f(x') - f(x'') | < \varepsilon \qquad \text{for } x', x'' \in [x_{i-1}, x_i], \qquad 1 \le i \le m.$$

Choose $\xi_i \in [x_{i-1}, x_i]$ and write

$$\Delta_i\phi = \phi(x_i) - \phi(x_{i-1}), \qquad \Delta_i\phi_n = \phi_n(x_i) - \phi_n(x_{i-1}).$$

By the mean value theorem for Stieltjes integrals,

$$\int_{x_{i-1}}^{x_i} f\, d\phi - f(\xi_i)\,\Delta_i\phi = [\, f(\xi'_i) - f(\xi_i)\,]\,\Delta_i\phi$$

for some $\xi'_i \in [x_{i-1}, x_i]$. Summing over i, we obtain

$$\left| \int_a^b f d\phi - \sum_{i=1}^m f(\xi_i)\,\Delta_i\phi \right| \le \sum_{i=1}^m \left| f(\xi'_i) - f(\xi_i) \right| \Delta_i\phi < \varepsilon \sum_{i=1}^m \Delta_i\phi \le \varepsilon M.$$

In the same way we find

$$\left| \int_a^b f\, d\phi_n - \sum_{i=1}^{m} f(\xi_i)\, \Delta_i \phi_n \right| < \varepsilon M.$$

Therefore,

$$\left| \int_a^b f\, d\phi - \int_a^b f\, d\phi_n \right| \leq \left| \int_a^b f\, d\phi - \sum_{i=1}^{m} f(\xi_i)\, \Delta_i \phi \right| + \left| \sum_{i=1}^{m} f(\xi_i)\, (\Delta_i \phi - \Delta_i \phi_n) \right|$$

$$+ \left| \int_a^b f\, d\phi_n - \sum_{i=1}^{m} f(\xi_i)\, \Delta_i \phi_n \right|$$

$$< 2M\varepsilon + \sum_{i=1}^{m} \left| f(\xi_i) \right| \left| \Delta_i(\phi - \phi_n) \right| .$$

Keeping the partition P_ε fixed, we have that the $\lim_{n \to \infty} \Delta_i(\phi - \phi_n) = 0$, hence

$$\limsup_{n \to \infty} \left| \int_a^b f\, d\phi - \int_a^b f\, d\phi_n \right| \leq 2M\varepsilon$$

and the desired conclusion follows.

Q.E.D.

In the next section we will see that with the aid of Helly's Selection Principle, we can find representative in the form of Stieltjes integral for a positive-definite moment functional L.

1.5 A Representation Theorem

Definition 1.5.1 Two distribution functions ψ_1 and ψ_2 are said to be *substantially equal* if and only if there is a constant C such that $\psi_1(x) = \psi_2(x) + C$ at all the points of continuity.

It is clear that substantially equal distribution functions have the same spectrum.

Let L be a positive-definite moment functional with moment sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$. According to the Gauss quadrature formula (Theorem 1.3.6), for each positive integer n, there are positive numbers $A_{n1}, \ldots, A_{nn}$ such that

$$L[x^k] = \mu_k = \sum_{i=1}^{n} A_{ni} x_{ni}^k, \qquad k = 0, 1, \ldots, 2n-1, \qquad (1.5.1)$$

where $x_{n1} < x_{n2} < \ldots < x_{nn}$ are the zeros of $P_n(x)$, the nth degree monic orthogonal polynomial corresponding to L.

Let ψ_n be defined by

$$\psi_n(x) = \begin{cases} 0 & \text{if } x < x_{n1} \\ A_{n1} + \ldots + A_{np} & \text{if } x_{n,p} \le x < x_{n,p+1} \quad (1 \le p < n) \\ \mu_0 & \text{if } x \ge x_{nn} \end{cases} \qquad (1.5.2)$$

ψ_n is a bounded, right continuous, non-decreasing step function whose spectrum is the finite set $\{x_{n1}, \ldots, x_{nn}\}$, and whose jump at x_{ni} is $A_{ni} > 0$. That is, ψ_n is a distribution function. Thus,

$$\int_{-\infty}^{\infty} x^k \, d\psi_n(x) = \sum_{i=1}^{n} A_{ni} x_{ni}^k = \mu_k, \qquad k = 0, 1, \ldots, 2n-1. \quad (1.5.3)$$

According to Theorem 1.4.2, $\{\psi_n\}$ contains a subsequence which converges on $(-\infty, \infty)$ to a bounded, non-decreasing function ψ. if $[\xi_1, \eta_1]$ is bounded, then Theorem 1.4.3 could be invoked to conclude from (1.5.3) that

$$\int_{-\infty}^{\infty} x^k \, d\psi(x) = \mu_k = L[x^k], \qquad k = 0, 1, 2, \ldots \qquad (1.5.4)$$

(note that $\psi(x) = 0$ for $x \leq \xi_1$ and $\psi(x) = \mu_0$ for $x \geq \eta_1$ so (1.5.4) can be written with the interval of integration reduced to $[\xi_1, \eta_1]$).

Theorem 1.5.2 Let L be a positive-definite moment functional and let ψ_n be defined by (1.5.2). Then there is a subsequence of $\{\psi_n\}$ that converges on $(-\infty, \infty)$ to a distribution function ψ which has an infinite spectrum and for which (1.5.4) is valid.

Proof: As already noted, there is a subsequence $\{\psi_{n(i)}\}$ which converges on $(-\infty, \infty)$ to a distribution function ψ. Writing $\phi_i = \psi_{n(i)}$, we have according to (1.5.3)

$$\int_{-\infty}^{\infty} x^k \, d\,\phi_i(x) = \mu_k \qquad \text{for} \quad n_i \geq \frac{k+1}{2} .$$

From Theorem 1.4.3 we conclude that for every compact interval $[\alpha, \beta]$,

$$\lim_{i\to\infty} \int_{\alpha}^{\beta} x^k \, d\,\phi_i(x) = \int_{\alpha}^{\beta} x^k \, d\,\psi(x) . \qquad (1.5.5)$$

Choosing $\alpha < 0 < \beta$ and $n_i > k + 1$, we can write

$$\left| \mu_k - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right| = \left| \int_{-\infty}^{\infty} x^k \, d\,\phi_i(x) - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right|$$

$$\leq \left| \int_{-\infty}^{\alpha} x^k \, d\,\phi_i(x) \right| + \left| \int_{\beta}^{\infty} x^k \, d\,\phi_i(x) \right| + \left| \int_{\alpha}^{\beta} x^k \, d\,\phi_i(x) - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right| .$$

But

$$\left| \int_{\beta}^{\infty} x^k \, d\,\phi_i(x) \right| = \left| \int_{\beta}^{\infty} \frac{x^{2k+2}}{x^{k+2}} \, d\,\phi_i(x) \right| \leq \beta^{-(k+2)} \left| \int_{\beta}^{\infty} x^{2k+2} \, d\,\phi_i(x) \right| \leq \beta^{-(k+2)} \mu_{2k+2} .$$

Similarly,

$$\left| \int_{-\infty}^{\alpha} x^k \, d\,\phi_i(x) \right| \le |\alpha|^{-(k+2)} \mu_{2k+2} ,$$

so that

$$\left| \mu_k - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right| \le \left| \int_{\alpha}^{\beta} x^k \, d\,\phi_i(x) - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right| + (|\alpha|^{-k-2} + \beta^{-k-2})\mu_{2k+2} .$$

Hence letting $i \to \infty$, we have by (1.5.5)

$$\left| \mu_k - \int_{\alpha}^{\beta} x^k \, d\,\psi(x) \right| \le (|\alpha|^{-k-2} + \beta^{-k-2})\,\mu_{2k+2} .$$

Now if we let $\alpha \to -\infty$ and $\beta \to \infty$, we obtain (1.5.4) .

Finally it is easy to show that ψ has an infinite spectrum. For if the spectrum of ψ consisted of exactly N points, we could construct a real polynomial $\pi(x)$ which vanished at these N points. We would then have

$$L[\pi^2(x)] = \int_{-\infty}^{\infty} \pi^2(x) \, d\,\psi(x) = 0 ,$$

contradicting the positive-definiteness of L.

Q.E.D.

We have thus shown that every positive-definite moment functional can be represented as a Stieltjes integral with a non-decreasing integrator ψ whose spectrum is an infinite set. We will say that "ψ provides a representation for L" or simply that "ψ is a representative of L".

Lemma 1.5.3 The spectrum of a distribution function is a closed set.

Proof: Let $x_n \in \sigma(\psi)$ $n = 0, 1, 2, \ldots$, and $\lim_{n\to\infty} x_n = x_0$. If $x_0 \notin \sigma(\psi)$, then there exists $\delta_0 > 0$ such that $\psi(x_0+\delta_0) - \psi(x_0-\delta_0) \le 0$. The fact ψ is non-decreasing forces $\psi(x) \equiv 0$ for $x \in (x_0-\delta_0, x_0+\delta_0)$. On the other hand, x_0 is a limit point of $\{x_n\}$ implies $x_n \in (x_0-\delta_0, x_0+\delta_0)$ when n is sufficient large. This yields $\psi(x_n+\delta_0) = \psi(x_n-\delta_0)$ which contradict the fact that $x_n \in \sigma(\psi)$.

Q.E.D.

Lemma 1.5.4 Let L be a positive-definite moment functional. Then L is positive-definite on $\sigma(\psi)$.

Proof: Let $\pi(x)$ be a polynomial which is non-negative on $\sigma(\psi)$ and does not vanish identically. Since $\sigma(\psi)$ is an infinite set, so there exists a $x_0 \in \sigma(\psi)$ such that $\pi(x_0) > 0$. By lemma 1.5.3 we know that $\sigma(\psi)$ is a closed set. Therefore x_0 is either an isolated point or an interior point of $\sigma(\psi)$. In each case, it is easy to see that

$$L[\,\pi(x)\,] = \int_{-\infty}^{\infty} \pi(x)\, d\psi(x) > 0 \ .$$

Therefore L is positive-definite on $\sigma(\psi)$.

Q.E.D.

Theorem 1.5.5 Every positive-definite moment functional L has a representative whose spectrum is a subset of $[\xi_1, \eta_1]$. Further, $[\xi_1, \eta_1]$ is a subset of every closed interval that contains the spectrum of some representative of L .

Proof: Let ψ_n be defined by (1.5.2) and let ψ denote the corresponding representative for L which is a subsequential limit of $\{\psi_n\}$. It is then clear that if $\xi_1 > -\infty$, $\psi(x) = 0$ for $x \le \xi_1$ while if $\eta_1 < \infty$, then $\psi(x) = \mu_0$ for $x \ge \eta_1$.

Thus the first assertion in the statement of the theorem follows. Now if ϕ is a representative for L and $\sigma(\phi) \subseteq [a, b]$, then by Lemma 1.5.4 and Theorem 1.3.1, L is positive-definite on [a, b]. By Theorem 1.3.2, $[\xi_1, \eta_1] \subseteq [a, b]$.

Q.E.D.

Definition 1.5.6 A positive-definite moment functional L is called *determinate* if any two representatives of L are substantially equal. (That is, L has a substantially unique representative.) Otherwise, L is called *indeterminate* .

Definition 1.5.7 A moment problem is called *determinate* if the corresponding moment functional is determinate. Otherwise, it is said to be an *indeterminate* moment problem.

1.6 Hamburger Moment Problem

We state the Hamburger moment problem in the following manner:

Given a real sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, find a distribution function ψ with an infinite spectrum, such that

$$\int_{-\infty}^{\infty} x^n \, d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \ldots . \qquad (1.6.1)$$

A real sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a *Hamburger moment sequence* , if it satisfies (1.6.1) .

Addressing only the simpler existence question for the Hamburger moment problem, we see in view of Theorem 1.5.2 that a necessary and sufficient condition that there is a distribution function ψ is that the corresponding moment functional L, defined by $L[x^n] = \mu_n$ be positive-definite. Combining this observation with Theorem 1.2.8, we can prove the following theorem, due to Hamburger [1].

Theorem 1.6.1 A necessary and sufficient condition that the Hamburger moment problem has a solution is that

$$\Delta_n\{1\} > 0 , \qquad n = 0, 1, 2, \dots . \qquad (1.6.2)$$

Proof: Firstly, we prove that the Hamburger moment problem has a solution if and only if L is positive-definite.

Assume that the Hamburger moment problem has a solution. That is, there exists a distribution function ψ with an infinite spectrum, such that

$$\int_{-\infty}^{\infty} x^n \, d\,\psi(x) = \mu_n .$$

So, for any polynomial $\pi(x)$ which is non-negative and does not vanish identically on **R**, we have

$$L[\pi(x)] = \int_{-\infty}^{\infty} \pi(x) \, d\,\psi(x) > 0 .$$

Therefore, L is positive-definite.

Conversely, if L is positive-definite, by Theorem 1.5.2 we know that the Hamburger moment problem has a solution.

Secondly, by Theorem 1.2.8, we have L is positive-definite if and only if $\Delta_n\{1\} > 0$ ($n = 0, 1, 2, \dots$).

Q.E.D.

Chapter 2

Some Representation Theorems

2.1 Preliminaries

In Chapter 1, we proved a representation theorem for a positive-definite moment functional based on Helly's theorems. With the aid of this representation theorem we obtain the characterization for the existence of a solution for the Hamburger moment problem that was first obtained by Hamberger [1]. In order to consider a similar question for different moment problems we need a representation theorem for a moment functional that is positive-definite on E, where E is the union of finite number of disjoint closed intervals.

In Section 2.2 we give two representation theorems. The first is due to M. Riesz [1]. It provides a method to construct a distribution function for a non-negative-definite moment functional. In order to use M. Riesz's method, we need a theorem for extending a positive-definite moment functional. First we define the unit step function at t, denoted by g(x;t), by

$$g(x;t) = \begin{cases} 1 & \text{if} \quad x \leq t \\ 0 & \text{if} \quad x > t \end{cases} . \qquad (2.1.1)$$

Let **R**[x] be the usual algebra of all real polynomials in the indeterminate x over the field of real numbers **R**. Let **R***[x] be the algebraic dual of **R**[x]. That is, **R***[x] is the set of all linear functionals L: **R**[x]→ **R**.

Let **S**[x] be the vector space of piecewise constant functions that are continuous from the left on **R** and tend to zero as $x \to \infty$. Let

$$\mathbf{G}[x] = \mathbf{S}[x] \oplus \mathbf{R}[x].$$

That is, **G**[x] is the vector space formed by taking the direct sum of polynomials and piecewise constant functions. Then we have the following theorem:

Theorem 2.1.1 (Akhiezer [3] P.69) Let E be the union of finite number of disjoint closed intervals and let $L \in \mathbf{R}^*[x]$ be a moment functional that is positive-definite on E. There exists a functional $\underline{L}$: $\mathbf{G}[x] \rightarrow \mathbf{R}$ such that for all polynomials $\pi(x) \in \mathbf{R}[x]$, $L[\pi(x)] = \underline{L}[\pi(x)]$ and $\underline{L}$ is non-negative-definite on E.

Proof: We denote by $g_1(x)$ any element of the space **G**[x] which does not belong to **R**[x] and we introduce the linear space $\mathbf{R}_1[x]$ of elements

$$f_1(x) = \pi(x) + \alpha g_1(x),$$

where $\pi(x)$ traverses **R**[x] and α traverses the set of all real numbers. We extend the functional L to L_1: $\mathbf{R}_1[x] \rightarrow \mathbf{R}$ by putting

$$L_1[f_1(x)] = L[\pi(x)] + \alpha r_1.$$

No matter how we choose the number r_1, which evidently represents $L_1[g_1(x)]$, the functional L_1 we have defined will be additive and homogeneous. Our problem consists in making the appropriate choice of the number r_1 such that the extended functional L_1 is non-negative-definite on E. For this purpose we take the set N_1^+ of all those polynomials $\pi(x) \in \mathbf{R}[x]$ for which $\pi(x) - g_1(x) \geq 0$ on E (It is easy to see that the set N_1^+ is not empty). Then we put

$$B_1 = \inf_{\pi(x) \in N_1^+} L[\pi(x)].$$

Further we introduce the set N_1^- (which is also non-empty) of those polynomials $\pi(x) \in \mathbf{R}[x]$ for which $g_1(x) - \pi(x) \geq 0$ on E and we put

$$b_1 = \sup_{\pi(x) \in N_1^-} L[\pi(x)] .$$

By virtue of these definitions $b_1 > -\infty$ and $B_1 < \infty$. We now prove that

$$b_1 \leq B_1 .$$

For any polynomials $\pi_1(x) \in N_1^+$ and $\pi_2(x) \in N_1^-$. We have $\pi_1(x) - g_1(x) \geq 0$ on E and $g_1(x) - \pi_2(x) \geq 0$ on E; This implies $\pi_1(x) - \pi_2(x) = (\pi_1(x) - g_1(x)) + (g_1(x) - \pi_2(x)) \geq 0$ on E. We now apply the functional to the polynomials $\pi_1(x)$, $\pi_2(x)$ and $\pi_1(x) - \pi_2(x)$. Therefore,

$$L[\pi_1(x)] - L[\pi_2(x)] = L[\pi_1(x) - \pi_2(x)] > 0,$$

whence

$$L[\pi_1(x)] > L[\pi_2(x)].$$

Since this inequality will hold for any $\pi_1(x) \in N_1^+$ and any $\pi_2(x) \in N_1^-$ we have that

$$\inf_{\pi_1(x) \in N_1^+} L[\pi_1(x)] \geq \sup_{\pi_2(x) \in N_1^-} L[\pi_2(x)],$$

i.e.

$$B_1 \geq b_1 .$$

We take for r_1 any number which satisfies the inequality $b_1 \leq r_1 \leq B_1$. We now prove that the functional L_1, defined on $\mathbf{R}_1[x]$ is non-negative-definite on E.

Let $f_1(x) \in \mathbf{R}_1[x]$ such that $f_1(x)$ is non-negative on E, then $f_1(x) = \pi(x) + \alpha g_1(x)$ where $\pi(x) \in \mathbf{R}[x]$ and $\alpha \in \mathbf{R}$.

If $\alpha = 0$, then $f_1(x) \in \mathbf{R}[x]$. It is trivial to show that $L_1[f_1(x)] \geq 0$.

For $\alpha > 0$, we have $f_1(x) = \pi(x) + \alpha g_1(x) = \alpha[g_1(x) - (-\pi(x)/\alpha)]$. $f_1(x) \geq 0$ on E implies $-\pi(x)/\alpha \in N_1^-$. Therefore,

$$L_1[f_1(x)] = L[\alpha(g_1(x) - (-\pi(x)/\alpha))] = \alpha(r_1 - L[-\pi(x)/\alpha]) \geq \alpha(r_1 - b) \geq 0.$$

In the case of $\alpha < 0$, we have $f_1(x) = \pi(x) + \alpha g_1(x) = -\alpha[(-\pi(x)/\alpha) - g_1(x)]$. $f_1(x) \geq 0$ implies $-\pi(x)/\alpha \in N_1^+$. Therefore

$$L_1[f_1(x)] = L[-\alpha((-\pi(x)/\alpha) - g_1(x))] = -\alpha(L[-\pi(x)/\alpha] - r_1) \geq -\alpha(B_1 - r_1) \geq 0.$$

Thus, L_1 is non-negative-definite on E.

The above described construction is the first step; it is followed by analogous further steps, and with the aid of (possibly transfinite) induction the functional thus obtained is extended to the whole space $\mathbf{G}[x]$ in such a way so that it is non-negative-definite on E.

Q.E.D.

In Section 2.2 we show how Theorem 2.1.1 can be used to find a representation theorem for a positive-definite moment functional. In the case when E is a compact set, we give another proof of the representation theorem for a positive-definite moment functional. To do this, we need the Hahn-Banach theorem and the F. Riesz representation theorem. Since these theorems are well known in functional analysis, we only state them without proof.

Let C[a, b] be the linear space of all the continous functions on [a, b], then we have the following theorems:

Theorem 2.1.2 (Hahn-Banach) (Mukherjea & Pothoven [1] P.249) Let P be a functional on C[a, b] satisfying: (i) $P[\alpha f(x)] = \alpha P[f(x)]$, (ii) $P[f(x) + g(x)]$

$\leq P[f(x)] + P[g(x)]$. If L is a linear functional on **R**[x] and $| L[f(x)] | \leq P[f(x)]$, $f(x) \in$ **R**[x], then there exists a linear functional $\underline{L}$ on C[a, b] such that

$$\underline{L}[f(x)] = L[f(x)] \quad f(x) \in \mathbf{R}[x] \quad \text{and} \quad | \underline{L}[f(x)] | \leq P[f(x)] \quad f(x) \in C[a, b] .$$

Theorem 2.1.3 (F. Riesz) (Mukherjea & Pothoven [1] P.282) Every non-negative-definite linear functional P on C[a, b] can be represented by

$$P[f(x)] = \int_a^b f(x)\, d\psi(x)$$

where $\psi(x)$ is an non-decreasing function.

2.2 Representation Theorems for Positive-definite Moment Functionals

Let E be a finite union of disjoint closed intervals. We now use M. Riesz method to construct a representative for a positive-definite moment functional on E. M. Riesz [1] originally discussed the case when $E = (-\infty, \infty)$. Akhiezer [3] provided the proof of the case when $E = [0, \infty)$. We now prove the general theorem.

Theorem 2.2.1 (M. Riesz) Let $\{\alpha_i \mid i = 1, ..., 2m\}$ be a finite sequence of real numbers such that $-\infty \leq \alpha_1 < \alpha_2 < ... < \alpha_{2m} \leq \infty$ and let $E = (\cup_{1 \leq i \leq m}[\alpha_{2i-1}, \alpha_{2i}]) \cap (-\infty, \infty)$. There exists a distribution function $\psi(x)$ such that

$$L[x^n] = \int_E x^n\, d\psi(x) \qquad \text{for} \qquad n = 0, 1, 2, ... ,$$

if and only if L is positive-definite on E.

Proof: Let $\psi(x)$ be a solution for the moment problem on E. Then for any polynomial $\pi(x)$ which is non-negative on E and does not vanish identically. we have

$$L[\pi(x)] = \int_E \pi(x)\, d\psi(x) > 0 \ .$$

Therefore, L is positive-definite on E.

Conversely, suppose L is positive-definite on E. Let us take any denumerable point set $S = \{t_k\}$ everywhere dense in E. We may evidently assume that $\alpha_i \in S$ (i=2, ..., 2m-1); if $-\infty < \alpha_1$, then $\alpha_1 \in S$ and if $\alpha_{2m} < \infty$, then $\alpha_{2m} \in S$.

We know that L is positive-definite on E. With the aid of Theorem 2.1.1, we extend this functional L to $\underline{L}$ defined on **G**[x]. We know that $\underline{L}$ is non-negative-definite on E. Define $\psi(t)$ by the following equation

$$\underline{L}\,[g_t(x)] = \psi(t)$$

where $g_t(x)$ is defined by (2.1.1).

In this way, we obtain a function $\psi(t)$ defined on S. We will show that this $\psi(t)$ is a solution of the moment problem. We first prove that this function is monotonic non-decreasing on S. Indeed, by virtue of the additivity, homogeneity and non-negativity of the extended functional $\underline{L}$, we can write down the equation

$$\psi(t'') - \psi(t') = \underline{L}\,[g_{t''}(x)] - \underline{L}\,[g_{t'}(x)] = \underline{L}\,[(g_{t''}(x) - g_{t'}(x))]$$

Also, since

$$g_{t''}(x) - g_{t'}(x) = \begin{cases} 0 & x \le t' \\ 1 & t' < x \le t'' \\ 0 & x > t'' \end{cases} ,$$

therefore

$$\psi(t'') - \psi(t') \geq 0$$

As a result of the monotonic property of $\psi(t)$ and the fact that the set S is dense in E the function $\psi(t)$ can be extended essentially uniquely to **R**, with its monotonic character preserved, by letting

$$\psi(x) = \begin{cases} \sup\{\ \psi(t) \mid t < x,\ t \in S\ \} & \text{if } x \geq \alpha_1 \\ \psi(\alpha_1) & \text{if } x < \alpha_1 \end{cases} .$$

To complete the proof we need only show that for any non-negative integer n

$$L[x^n] = \int_E x^n \, d\psi(x) \ .$$

Let B be any real number > $\max\{|\alpha_2|, |\alpha_{2m-1}|, 1\}$ and let ε be any positive number .

Define a set of points P_{mN}, by $P_{mN} = \{\tau_{i,j} \mid i \leq i \leq m,\ 0 \leq j \leq N_i\}$ $(N = N_1 + \dots + N_m)$, such that

(i) $\tau_{i,j} \in [\alpha_{2i-1}, \alpha_{2i}]$ for $0 \leq j \leq N_i$, $1 \leq i \leq m$ and $P_{mN} \subseteq S$;

(ii) $\tau_{i,0} = \alpha_{2i-1}$ $i = 2, \dots, m$ and $\tau_{i,Ni} = \alpha_{2i}$ $i = 1, \dots, m-1$;

(iii) $\tau_{1,0} = \begin{cases} -B & \text{if } -\infty = \alpha_1 \\ \alpha_1 & \text{if } -\infty < \alpha_1 \end{cases}$ and $\tau_{m,N_m} = \begin{cases} B & \text{if } \alpha_{2m} = \infty \\ \alpha_{2m} & \text{if } \alpha_{2m} < \infty \end{cases}$;

(iv) $|(\tau_{i,j+1})^n - (\tau_{i,j})^n| < \varepsilon$ for $1 \leq i \leq m$, $0 \leq j \leq N_i$.

Now define the function

$$F^{n}_{mN}(x) = \sum_{i=1}^{m} \sum_{j=0}^{N_i-1} (\tau_{i,j})^n [\, g(x;\tau_{i,j+1}) - g(x;\tau_{i,j})\,].$$

It is not difficult to verify that for all x belonging to E,

$$-x^s/B - \varepsilon \le x^n - F_{mN}{}^n(x) \le \varepsilon + x^s/B,$$

where s equal to the smallest even integer greater than n. We now apply the linear functional $\underline{L}$ to this inequality to obtain

$$-\frac{L[\,x^s]}{B} - \varepsilon L[\,1\,] \le L[\,x^n] - \sum_{i=1}^{m}\sum_{j=0}^{N_i-1}(\tau_{i,j})^n[\,\psi(\tau_{i,j+1}) - \psi(\tau_{i,j})\,] \le \varepsilon L[\,1\,] + \frac{L[\,x^s]}{B}\,.$$

Now by using the fact that the summation in this expression is a Stieltjes sum, we obtain by letting $\|\,\Delta\tau\,\| \to 0$ ($\|\,\Delta\tau\,\| = \max|\,\tau_{i,j+1} - \tau_{i,j}\,|$)

$$-\frac{\mu_s}{B} \le \mu_n - \int_{\tau_{1,0}}^{\alpha_2} x^n\, d\psi(x) - \int_{\alpha_{2m-1}}^{\tau_{m,Nm}} x^n\, d\psi(x) - \sum_{i=2}^{m-1}\int_{\alpha_{2i-1}}^{\alpha_{2i}} x^n\, d\psi(x) \le \frac{\mu_s}{B}\,.$$

Finally, if we let $B \to \infty$ we obtain the required result

$$L[\,x^n\,] = \mu_n = \int_E x^n d\psi(x) \quad .$$

Q.E.D.

In the case when the true interval of orthogonality (see Definition 1.3.5) is a compact set, we have another approach for obtaining a representative of a positive-definite moment functional. It uses the Hahn-Banach Theorem and the F. Riesz representation theorem as given in Section 2.1.

Theorem 2.2.2 Let L be a positive-definite moment functional whose true interval of orthogonality, $[\xi_1, \eta_1]$, is a compact set. Then there is a distribution function ψ with an infinite spectrum contained in $[\xi_1, \eta_1]$, such that for any polynomial $\pi(x)$

$$L[\pi(x)] = \int_{\zeta_1}^{\eta_1} \pi(x)\, d\,\psi(x).$$

Proof: Let L be a positive-definite whose true interval of orthogonality $[\xi_1, \eta_1]$ is compact.

First, we extend this functional L to the linear space $C[\xi_1, \eta_1]$. To do this we proceed as follows. For any $t(x) \in C[\xi_1, \eta_1]$, let $P[t(x)] = \mu_0 \max\{|t(x)| \mid \xi_1 \le x \le \eta_1\}$ where $\mu_0 = L[1] > 0$. Then it is easy to see that (i) $P[\alpha t(x)] = \alpha P[t(x)]$ and (ii) $P[t_1(x) + t_2(x)] \le P[t_1(x)] + P[t_2(x)]$, for any $t_1(x), t_2(x) \in C[\xi_1, \eta_1]$.

Let $\pi(x)$ be a polynomial of degree n, we can write it as

$$\pi(x) = \sum_{k=0}^{n} a_k x^k$$

By using Gauss quadrature formula (Theorem 1.3.6), we have

$$L[\pi(x)] = \sum_{k=0}^{n} \pi(x_{nk})\, A_{nk} \le \sum_{k=0}^{n} \max_{\zeta_1 \le x \le \eta_1}\{|\pi(x)|\}\, A_{nk}$$
$$= \max_{\zeta_1 \le x \le \eta_1}\{|\pi(x)|\} \sum_{k=0}^{n} A_{nk} = \mu_0 \max_{\zeta_1 \le x \le \eta_1}\{|\pi(x)|\} = p[\pi(x)]$$

i.e. $L[\pi(x)] \le P[\pi(x)]$. Similarly, we can prove $L[\pi(x)] \ge -P[\pi(x)]$. Therefore the following inequality is true:

$$|\, L[\pi(x)]\,| \le P[\pi(x)].$$

According to Hahn-Banach Theorem (Theorem 2.1.2), there is a linear functional $\underline{L}$ defined on $C[\xi_1, \eta_1]$ which is an extension of L.

Next, we prove $\underline{L}$ is a non-negative-definite functional. For any $f(x) \in C[\xi_1,$

$\eta_1]$ which is non-negative on $[\xi_1, \eta_1]$. By Weierstrass Theorem (Apostol [1] P.481), there exists a sequence of polynomials $\{Q_n(x)\}$ converges uniformly to f(x). Let $\rho_n = \max\{| Q_n(x) - f(x)| \mid \xi_1 \le x \le \eta_1\}$ ($\rho_n \to 0$ when $n \to \infty$), then $\pi_n(x) = Q_n(x) + \rho_n$ are polynomials and $\pi_n(x) \ge f(x)$ for n =0, 1, 2, ... Moreover, $\pi_n(x)$ converges to f(x) uniformly. So, given $\varepsilon > 0$, the following is true:

$$\left|\pi_n(x) - f(x)\right| < \frac{\varepsilon}{2\mu_0} \qquad \text{for} \quad x \in [\zeta_1, \eta_1]$$

when n is sufficient large. Since we also have

$$| \underline{L}[\pi_n(x) - f(x)] | \le \mu_0 \max_{\xi 1 \le x \le \eta 1}\{ | \pi_n(x) - f(x)| \} .$$

It is easy to see that $\lim_{n \to \infty} | \underline{L}[\pi_n(x) - f(x)] | = 0$. This implies that the

$\lim_{n \to \infty} \underline{L}[\pi_n(x) - f(x)] = 0$. So, we have $\lim_{n \to \infty} L[\pi_n(x)] = \underline{L}[f(x)]$.

By the positivity of L , we have $\underline{L}[f(x)] = \lim_{n \to \infty} L[\pi_n(x)] \ge 0$. Therefore $\underline{L}$ is a non-negative-definite functional.

Finally, we note that, the required result follows directly from F. Riesz Theorem (Theorem 2.1.3).

Q.E.D.

Theorem 2.2.2 is posed as a problem in Chihara's text (Chihara [1] P.59).

2.3 Representation Theorems for Polynomials

Akhiezer [3] shows that if p(x) is a polynomial of degree n then (i) if $p(x) \ge 0$ on [0, 1], then $p(x) = x[A_m(x)]^2 + (1-x)[B_m(x)]^2$ for $n = 2m + 1$, and $p(x) = [C_m(x)]^2 + x(1-x)[D_{m-1}(x)]^2$ for $n = 2m$, and (ii) if $p(x) \ge 0$ on $[0, \infty)$, then

$p(x) = [A(x)]^2 + x[B(x)]^2$. The first representation can be used to find a characterization for the existence of a solution to the Hausdorff moment problem and the second one can be used for the Stieltjes moment problem. We wish to obtain the analogous representation for polynomials $p(x)$ that are bigger than or equal to zero on a set E where E is a finite union of disjoint closed intervals.

The following representation theorems for the polynomials which are non-negative on $(-\infty, a]\cup[b, \infty)$ will be used in later chapters.

Theorem 2.3.1 Any polynomial $\pi(x)$ of degree n which is non-negative and does not vanish identically on $(-\infty, a]\cup[b, \infty)$ can be written as

$$\pi(x) = A(x) + (x-a)(x-b)B(x)$$

where $A(x)$ is either a polynomial non-negative on **R** with degree at most n or $A(x) \equiv 0$; $B(x)$ is either a polynomial non-negative on **R** with degree at most n-2 or $B(x) \equiv 0$; $A(x) + B(x)$ is not equal to zero identically.

Proof: We note that, if a polynomial $\pi(x)$ with degree n is non-negative on $(-\infty, a]\cup[b, \infty)$, then n must be even and the leading coefficient of $\pi(x)$ must be positive. Therefore, we only need consider the monic and even degree polynomials.

If $n = 0$, then $\pi(x) = 1$. We choose $A(x) = 1$, $B(x) \equiv 0$. Obviously, the theorem is true.

For $n = 2$, we have $\pi(x) = x^2 + qx + r$. There are two cases to consider: First, if $q^2 - 4r \le 0$, we know that $\pi(x) \ge 0$ everywhere on **R**. We choose $A(x) = \pi(x)$, $B(x) \equiv 0$. The other case is when $q^2 - 4r > 0$. We have $\pi(x) = (x - x_1)(x - x_2)$. $\pi(x)$ is non-negative on $(-\infty, a]\cup[b, \infty)$ implies both x_1 and

$x_2 \in [a, b]$. We find that if we let

$$\begin{cases} p = \dfrac{x_1x_2 - ab \pm \sqrt{(b-x_1)(b-x_2)(x_1-a)(x_2-a)}}{a + b - x_1 - x_2} \\ k = \dfrac{[\sqrt{(b-x_1)(x_2-a)} \pm \sqrt{(b-x_2)(x_1-a)}\,]^2}{(b-a)^2} \end{cases}$$

then

$$\pi(x) = (x - x_1)(x - x_2) \equiv (1 - k)(x + p)^2 + k(x - a)(x - b)$$

Moreover, p is a real number and $0 \le k \le 1$. So, we can choose $B(x) = k$ and $A(x) = (1 - k)(x + p)^2$. Therefore, the theorem is true when $n = 2$.

Assume the theorem is true for all the even numbers less or equal to 2k.

Let $\pi(x)$ be a polynomial of degree 2k+2 that is non-negative on $(-\infty, a] \cup [b, \infty)$. By the Factorization Theorem (Archbold [1] P.129), we can write $\pi(x)$ in the following form:

$$\pi(x) = (x - a_1)^{k(1)} \ldots (x - a_i)^{k(i)}(x^2 + p_1x + q_1)^{r(1)} \ldots (x^2 + p_jx + q_j)^{r(j)}$$

(i) If there is a $k(s) \ge 2$ $(1 \le s \le i)$, then we choose $\pi_1(x) = (x - a_s)^2$ and $\pi_2(x) = \pi(x) / \pi_1(x)$.

(ii) If there is a $r(t) \ge 1$ $(1 \le t \le j)$, then we choose $\pi_1(x) = (x^2 + p_tx + q_t)$ and $\pi_2(x) = \pi(x) / \pi_1(x)$.

(iii) If (i) and (ii) do not apply, then we have

$$\pi(x) = \prod_{i=1}^{n} (x - x_i)$$

where all the x_i's are different. This implies $x_i \in [a, b]$ $(i = 1, \ldots n.)$. In this case we can choose any two zeros of $\pi(x)$, say x_i and x_j . Let $\pi_1(x) = (x - x_i)(x - x_j)$ and $\pi_2(x) = \pi(x) / \pi_1(x)$.

Therefore for each of the three cases above we can write $\pi(x)$ as the product of two polynomials $\pi_1(x)$ and $\pi_2(x)$. Both of them are non-negative on $(-\infty, a]\cup[b, \infty)$ and their degree is not more than 2k.

In order to use mathematical induction it remains to be noted that the equalities

$\pi_1(x) = A_1(x) + (x -a)(x -b)B_1(x)$ and $\pi_2(x) = A_2(x) + (x -a)(x -b)B_2(x)$

imply the equality

$$\pi_1(x)\, \pi_2(x) = [A_1(x)A_2(x) + (x-a)^2(x -b)^2B_1(x)B_2(x)] \\ + (x -a)(x -b)[A_1(x)B_2(x) + A_2(x)B_1(x)] .$$

Q.E.D.

In Chapter 4, we will use the above theorem to obtain a characterization for a complemented Hausdorff moment sequence. In order to obtain characterizations for a moment sequence on E where E is a finite union of disjoint closed intervals, we need a representation theorem for a polynomial that is non-negative on E.

Let $\{\alpha_i \mid i = 1, ..., 2m\}$ be a finite sequence of real numbers such that

$$-\infty \le \alpha_1 < \alpha_2 < ... < \alpha_{2m} \le \infty \quad \text{and} \quad E = (\cup_{1\le i \le m}[\alpha_{2i-1}, \alpha_{2i}])\cap(-\infty, \infty) . \quad (2.3.1)$$

Let $\mathbf{A} = \{ A_s(x) \mid A_s(x) = d_s\Pi_{i\in S}(x - \alpha_i), \mid \alpha_i \mid < \infty, S \subseteq \{1, ..., 2m\}, d_s = \pm1, A_s(x) \ge 0 \text{ for } x\in E \}$. By adopting the usual convention that the vacuous product is taken to be 1, we see that the constant polynomial 1 belongs to $\mathbf{A}$.

Proposition 2.3.2 If both $A_s(x)$ and $A_t(x) \in \mathbf{A}$, then $A_s(x)A_t(x) =$

$A_u(x)C(x)$ where $A_u(x) \in \mathbf{A}$ and $C(x)$ is a polynomial which is non-negative and does not vanish identically on $\mathbf{R}$.

Proof: Let $A_s(x), A_t(x) \in \mathbf{A}$, then

$$A_s(x)A_t(x) = d_s\Pi_{i\in S}(x - \alpha_i)d_t\Pi_{i\in T}(x - \alpha_i) = d_sd_t\Pi_{k\in S\cup T\setminus(S\cap T)}(x - \alpha_k)\Pi_{r\in S\cap T}(x - \alpha_r)^2$$

$$= d_u\Pi_{k\in U}(x - \alpha_k)C(x) = A_u(x)C(x)$$

where $d_u = d_sd_t$, $U = S\cup T\setminus(S\cap T)$ and $C(x) = \Pi_{r\in S\cap T}(x - \alpha_r)^2$. Therefore $A_u(x)\in \mathbf{A}$ and $C(x)$ is a polynomial which is non-negative and does not vanish identically on $\mathbf{R}$.

Q.E.D.

Let $\mathbf{P} = \{ \Sigma A_i(x)C_i(x) \mid A_i(x) \in \mathbf{A}, C_i(x) \in \mathbf{R}[x]$ and $C_i(x) \geq 0$ on $\mathbf{R} \}$. By Proposition 2.3.2, it is obvious that the following proposition is true.

Proposition 2.3.3 $\mathbf{P}$ is closed with respect to ordinary polynomial addition and multiplication.

Now we give the representation theorem for the polynomials that are non-negative on E where E is defined by (2.3.1).

Theorem 2.3.4 Let $\pi(x)\in \mathbf{R}[x]$ and E be a subset of $\mathbf{R}$ defined by (2.3.1). $\pi(x) \geq 0$, for $x \in E$, if and only if $\pi(x)\in \mathbf{P}$.

Proof: From the definition of $\mathbf{P}$, it is obvious that if $\pi(x)\in \mathbf{P}$, then $\pi(x) \geq 0$, for $x\in E$.

Conversely, let $\pi(x) \geq 0$, for $x \in E$. There are four cases to consider. Namely, (i) $-\infty = \alpha_1$, $\alpha_{2m} = \infty$; (ii) $-\infty < \alpha_1$, $\alpha_{2m} = \infty$; (iii) $-\infty = \alpha_1$, $\alpha_{2m} < \infty$; (iv) $-\infty < \alpha_1$, $\alpha_{2m} < \infty$. For sake of argument let us assume that we have case (iii). That is, $-\infty = \alpha_1 < \alpha_2 < \ldots < \alpha_{2m} < \infty$. The other cases can be handled in a similar manner.

Let $D = \{a_1, a_2, \dots, a_r\}$ be the set of all the distinct real zeros of $\pi(x)$. Then

$$\pi(x) = d\prod_{i=1}^{r}(x - a_i)^{s_i} D(x), \quad \text{where } D(x) \geq 0 \text{ on } \mathbf{R} \text{ and } d = \pm 1.$$

Let us partition **R** into the following three sets: $T_1 = [\alpha_{2m}, \infty)$

$T_2 = \cup_{1 \leq i \leq m}(\alpha_{2i-1}, \alpha_{2i})$ and $T_3 = \cup_{1 \leq i \leq m-1}[\alpha_{2i}, \alpha_{2i+1}]$. Then,

$$\pi(x) = d\prod_{a_i \in T_1}(x - a_i)^{s_i}\prod_{a_j \in T_2}(x - a_j)^{s_j}\prod_{a_k \in T_3}(x - a_k)^{s_k} D(x). \qquad (2.3.2)$$

If there are even number of zeros of $\pi(x)$ in T_1, then $d = 1$ and

$$d\prod_{a_i \in T_1}(x - a_i)^{s_i} \in \mathbf{P}.$$

If there are odd number of zeros of $\pi(x)$ in T_1, then $d = -1$. Let $a_r \in T_1$, then $a_r - x = (a_r - \alpha_{2m}) + d(x - \alpha_{2m})$ which is an element in **P**. Thus, by Proposition 2.3.3 we know that

$$d\prod_{a_i \in T_1}(x - a_i)^{s_i} \in \mathbf{P}.$$

Now consider what happens when one of the zeros of $\pi(x)$, call it a_r is in one of the sets T_2 or T_3.

If $a_r \in T_2$, then because $\pi(x)$ is non-negative on T_2 and because T_2 is an open set, we have that a_r is a root of $\pi(x)$ that must have even multiplicity. Therefore s_r is an even number . Thus

$$\prod_{a_j \in T_2} (x - a_j)^{s_j} \in \mathbf{P}.$$

Finially, let us consider what happens when $a_r \in T_3$. That is suppose for some s, $a_r \in [\alpha_{2s}, \alpha_{2s+1}]$, where $1 \leq s \leq m-1$. Because $\pi(x) \geq 0$, for all $x \in E$, therefore $\pi(x)$ has an even number of zeros in $[\alpha_{2s}, \alpha_{2s+1}]$. Thus , there must exist another zero, call it a_t such that $a_t \in [\alpha_{2s}, \alpha_{2s+1}]$. By Theorem 2.3.1 we have that $(x - a_r)(x - a_t) \in \mathbf{P}$ and therefore

$$\prod_{a_k \in T_3} (x - a_k)^{s_k} \in \mathbf{P}.$$

Thus each of the factors that make up $\pi(x)$ as given in (2.3.2) are elements of **P** and therefore by Proposition 2.3.3 we have that $\pi(x) \in \mathbf{P}$.

Q.E.D.

Chapter 3

The Classical Moment Problems

3.1 Dual of a Polynomial

Let $\pi(x)$ be a polynomial belong to **R**[x]. We define the dual of $\pi(x)$, denoted by $\{\pi(x)\}^*$, by $\{\pi(x)\}^* : \mathbf{R}^*[x] \to \mathbf{R}^*[x]$ and $\{\pi(x)\}^*L[p(x)] = L[\pi(x)p(x)]$, where juxtaposition of $\pi(x)$ and $p(x)$ on the right hand side of this last equation is ordinary polynomial multiplication.

In the literature the OPS corresponding to the moment functional $\{x\}^*L$ is called the "Kernel polynomial sequence".

With different choices of $\pi(x)$ we have the following theorems which will be used later.

Lemma 3.1.1 Let $E = \{x_{nk} \mid P_n(x_{nk}) = 0,\ k = 1, \dots, n;\ n = 1, 2, \dots\}$. If L is positive-definite, then L is positive-definite on E.

Proof: Given a polynomial $\pi(x)$ of degree m which is non-negative and does not vanish identically on E, we let $n = m + 1$. By using Gauss quadrature formula, we have

$$L[\pi(x)] = \sum_{k=1}^{n} A_{nk}\pi(x_{nk}) > 0 .$$

This inequality is true because $A_{nk} > 0$, $\pi(x_{nk}) \geq 0$ and at least one of the $\pi(x_{nk})$'s is positive.

Q.E.D.

Lemma 3.1.2 If L is positive-definite, then L is positive-definite on its true interval of orthogonality $[\xi_1, \eta_1]$.

Proof: From Lemma 3.1.1, we know that L is positive-definite on E. Since $E \subseteq [\xi_1, \eta_1]$, by using Theorem1.3.1 we have that L is positive-definite on $[\xi_1, \eta_1]$.

Q.E.D.

Theorem 3.1.3 (Chihara [1] P.36) Let $\{P_n(x)\}$ be an OPS for L. If κ is not a zero of $P_n(x)$ for all n, and L is positive-definite, then $\{x - \kappa\}^*L$ is also positive-definite if and only if $\kappa \leq \xi_1$.

Proof: If L is positive-definite and $-\infty < \kappa \leq \xi_1$, then for any polynomial $\pi(x)$ which is non-negative and does not vanish identically on **R**, we have that $(x - \kappa)\pi(x)$ is non-negative on $[\xi_1, \eta_1]$ and does not vanish identically. By Lemma 3.1.2, we know that L is positive-definite on $[\xi_1, \eta_1]$. Therefore

$$\{x - \kappa\}^*L[\pi(x)] = L[(x - \kappa)\pi(x)] > 0 .$$

So, $\{x - \kappa\}^*L$ is positive-definite.

Conversely, suppose that $\{x - \kappa\}^*L$ and L are both positive-definite. Put

$$\rho(x) = (x - x_{n1})^{-1}P_n(x) .$$

An application of the Gauss quadrature formula yields

$$0 < \{x - \kappa\}^*L[\rho^2(x)] = L[(x - \kappa)\rho^2(x)] = A_{n1}(x_{n1} - \kappa)\rho^2(x_{n1}).$$

That shows that $\kappa < x_{n1}$ and hence that $\kappa \leq \xi_1$.

Q.E.D.

We now want to prove the following analogue of Theorem 3.1.3. Its proof is analogous to the proof of Theorem 3.1.3.

Theorem 3.1.4 Let $\{P_n(x)\}$ be an OPS for L. If ω is not a zero of $P_n(x)$ for all n, and L is positive-definite, then $\{\omega - x\}^*L$ is also positive-definite if and only if $\omega \geq \eta_1$.

Proof: Let L be positive-definite and $\eta_1 \leq \omega < \infty$. Then for any polynomial $\pi(x)$ which is non-negative and does not vanish identically on **R**, we have $(\omega - x)\pi(x)$ is non-negative and does not vanish identically on $[\xi_1, \eta_1]$. We know that L is positive-definite on its true interval of orthogonality $[\xi_1, \eta_1]$ (Lemma 3.1.2). Therefore

$$\{\omega - x\}^*L[\pi(x)] = L[(\omega - x)\pi(x)] > 0 .$$

So, $\{\omega - x\}^*L$ is positive-definite.

Conversely, suppose $\{\omega - x\}^*L$ and L are positive-definite. Put

$$\rho(x) = (x - x_{nn})^{-1}P_n(x)$$

An application of the Gauss quadrature formula yields

$$0 < \{\omega - x\}^*L[\rho^2(x)] = L[(\omega - x)\rho^2(x)] = A_{nn}(\omega - x_{nn})\rho^2(x_{nn})$$

This shows that $\omega > x_{nn}$ and hence that $\omega \geq \eta_1$.

Q.E.D.

The next analogue of Theorem 3.1.3 uses both ends of the true interval of orthogonality.

Theorem 3.1.5 Let $\{P_n(x)\}$ be an OPS for L and let κ and ω be two real numbers such that $\kappa < \omega$. If neither κ nor ω are zeros of $P_n(x)$ for all n and L is positive-definite, then $\{(x - \kappa)(\omega - x)\}^*L$ is also positive-definite if and only if $\kappa \leq \xi_1 < \eta_1 \leq \omega$.

Proof: Let L be positive-definite and let $\kappa \le \xi_1 < \eta_1 \le \omega$. Thus we have from Theorem 1.3.1 and Lemma 3.1.2 that L is positive-definite on $[\kappa, \omega]$.

Let $\pi(x)$ be any polynomial which is non-negative and does not vanish identically on **R**. Then $(x - \kappa)(\omega - x)\pi(x)$ is non-negative on $[\kappa, \omega]$ and does not vanish identically. Thus, $\{(x - \kappa)(\omega - x)\}^*L[\pi(x)] = L[(x - \kappa)(\omega - x)\pi(x)] > 0$ and therefore $\{(x - \kappa)(\omega - x)\}^*L$ is positive-definite .

Conversely, let both L and $\{(x - \kappa)(\omega - x)\}^*L$ be positive-definite. We first show that $\{x - \kappa\}^*L$ is positive-definite. Let $\pi(x)$ be a polynomial which is non-negative and does not vanish identically on **R**. We have

$$\{\frac{(x - \kappa)}{(\omega - \kappa)}\}^*L\,[\pi(x)] = L\,[\frac{x - \kappa}{\omega - \kappa}(\frac{\omega - x}{\omega - \kappa} - \frac{\omega - x}{\omega - \kappa} + 1)\pi(x)]$$

$$= L[(\frac{x - \kappa}{\omega - \kappa})(\frac{\omega - x}{\omega - \kappa})\pi(x)] + L[(\frac{x - \kappa}{\omega - \kappa})(1 - \frac{\omega - x}{\omega - \kappa})\pi(x)]$$

$$= \{\frac{(x - \kappa)(\omega - x)}{(\omega - \kappa)^2}\}^*L\,[\pi(x)] + L\,[(\frac{x - \kappa}{\omega - \kappa})^2\pi(x)] > 0 .$$

Note that if c is a positive number, then L is positive-definite if and only if $\{c\}^*L$ is positive-definite. Therefore, $\{x - \kappa\}^*L$ is positive-definite and hence by Theorem 3.1.3 $\kappa \le \xi_1$.

In a similar manner, by using Theorem 3.1.4, we can show that $\eta_1 \le \omega$.

Q.E.D.

3.2 Stieltjes and Complemented Stieltjes Moment Problems

The *Stieltjes moment problem* is the following: Given a real sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, find a distribution function ψ with an infinite spectrum contained in $[0, \infty)$ such that

$$\int_0^\infty x^n \, d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \ldots . \qquad (3.2.1)$$

A real sequence will be called a *Stieltjes moment sequence* if it satisfies (3.2.1).

In order to find a characterization for the existence of a solution to the Stieltjes moment problem, we need the following theorems.

Theorem 3.2.1 The Stieltjes moment problem has a solution if and only if L is positive-definite on $[0, \infty)$.

Proof: If the Stieltjes moment problem has a solution, that is, there exists a distribution function ψ with an infinite spectrum contained in $[0, \infty)$, such that

$$\int_0^\infty x^n \, d\,\psi(x) = \mu_n ,$$

then, for any polynomial $\pi(x)$ which is non-negative and does not vanish identically on $[0, \infty)$, we have

$$L[\,\pi(x)\,] = \int_0^\infty \pi(x) \, d\,\psi(x) > 0 .$$

Therefore, L is positive-definite on $[0, \infty)$

Conversely, suppose that L is positive-definite on $[0, \infty)$. According to Theorem 1.3.2, we know that all the zeros of $P_n(x)$ belong to interval $[0, \infty)$, $n \geq 1$. This implies that $\xi_1, \eta_1 \in [0, \infty)$, i.e. $[\xi_1, \eta_1] \subseteq [0, \infty)$.

Therefore, by Theorem 1.5.5, there exists a distribution function $\psi(x)$ with an infinite spectrum contained in $[\xi_1, \eta_1] \subseteq [0, \infty)$, such that

$$\mu_n = L[x^n] = \int_0^\infty x^n \, d\,\psi(x) \ .$$

Q.E.D.

Theorem 3.2.2 L is positive-definite on $[0, \infty)$ if and only if both L and $\{x\}^*L$ are positive-definite.

Proof: If L is positive-definite on $[0, \infty)$, then by Theorem 1.3.1 we have L positive-definite.

We know that L is positive-definite on $[0, \infty)$ implies $[\xi_1, \eta_1] \subseteq [0, \infty)$. Hence we obtain that $\xi_1 \geq 0$.

Since L is positive-definite and $\xi_1 \geq 0$ we have by using Theorem 3.1.3 with $\kappa = 0$, that $\{x\}^*L$ is positive-definite.

Conversely, suppose that L and $\{x\}^*L$ are positive-definite. Again, by Theorem 3.1.3, we have $\xi_1 \geq 0$. With the aid of Lemma 3.1.2, we obtain L is positive-definite on $[0, \infty)$.

Q.E.D.

Theorem 3.2.3 L and $\{x\}^*L$ are positive-definite if and only if both $\Delta_n\{1\} > 0$ and $\Delta_n\{x\} > 0$ ($n = 0, 1, 2, \ldots$).

Proof: Theorem 1.2.8 indicates that L is positive-definite if and only if $\Delta_n\{1\} > 0$.

Let $\mu_n^* = \{x\}^*L[\, x^n \,]$. By the definition of $\{x\}^*L$, we have

$$\mu_n^* = \mu_{n+1} \qquad n = 0, 1, 2, \ldots \ .$$

By Theorem 1.2.8 we obtain that $\{x\}^*L$ is positive-definite if and only if $\Delta_n\{x\} > 0$.

Q.E.D.

Theorem 3.2.4 (Stieltjes [1]) A necessary and sufficient condition that the Stieltjes moment problem has a solution is that both

$$\Delta_n\{1\} > 0 \text{ and } \Delta_n\{x\} > 0, \quad \text{for} \quad n = 0, 1, 2, \ldots \,. \qquad (3.2.2)$$

Proof: Theorem 3.2.4 follows directly from Theorem 3.2.1, Theorem 3.2.2 and Theorem 3.2.3.

Q.E.D.

Using the same method as above we can prove the modified Stieltjes moment problem, which requires that the spectrum of the distribution function be contained in $[a, \infty)$ for some finite number a. We state the result as the following theorem without proof.

Theorem 3.2.5 Given a real sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, in order that there exists a distribution function ψ with an infinite spectrum contained in $[a, \infty)$ such that

$$\mu_n = \int_a^\infty x^n \, d\,\psi(x) \qquad n = 0, 1, 2, \ldots \,, \qquad (3.2.3)$$

it is necessary and sufficient that both

$$\Delta_n\{1\} > 0 \text{ and } \Delta_n\{x - a\} > 0, \quad \text{for} \quad n = 0, 1, 2, \ldots \,. \qquad (3.2.4)$$

In fact Theorem 3.2.5 can be obtain directly from Theorem 3.2.4 by a linear transformation of the independent variable.

Now let us consider the following moment problem where the corresponding spectrum is a subset of $(-\infty, b]$ where b is any finite number. That is, given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, find a necessary and sufficient conditions in order that there exists a distribution function ψ with an infinite spectrum contained in $(-\infty, b]$ such that

$$\mu_n = \int_{-\infty}^{b} x^n \, d\,\psi(x), \quad n = 0, 1, 2, \ldots . \qquad (3.2.5)$$

We call this moment problem the *complemented Stieltjes moment problem* .

A sequence $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is called a *complemented Stieltjes moment sequence* if it satisfies (3.2.5).

With the aid of Theorem 3.1.4, it is easy to find a characterization for the complemented Stieltjes moment sequence. We state the result without proof.

Theorem 3.2.6 A necessary and sufficient condition that the complemented Stieltjes moment problem has a solution is that both

$$\Delta_n\{1\} > 0 \text{ and } \Delta_n\{b - x\} > 0, \quad n = 0, 1, 2, \ldots . \qquad (3.2.6)$$

3.3 Hausdorff Moment Problem

The moment problem for the case of a finite interval [0, 1] is called Hausdorff's moment problem. We state it as the following: Given a real sequnece $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, find a distribution function ψ with an infinite spectrum contained in [0, 1] such that

$$\int_{0}^{1} x^n \, d\,\psi(x) = \mu_n, \qquad n = 0, 1, 2, \ldots . \qquad (3.3.1)$$

A real sequence $\{ \mu_n \mid n = 0, 1, 2, \ldots \}$ will be called a Hausdorff moment sequence, if it satisfies (3.3.1).

Theorem 3.3.1 (Akhiezer [3] P.74) A necessary and sufficient condition that the Hausdorff moment problem has a solution is that

$$\Delta_n\{1\} > 0, \qquad \Delta_n\{x\} > 0, \qquad \Delta_n\{1 - x\} > 0$$

$$\text{and} \quad \Delta_n\{x(1 - x)\} > 0 \quad \text{for} \quad n = 0, 1, 2, \dots . \qquad (3.3.2)$$

Proof: First, by using the representation theorem (either Theorem 1.5.5 or Theorem 2.2.2) we know that Hausdorff moment problem has a solution if and only if L is positive-definite on [0, 1] where $L[x^n] = \mu_n$ for $n = 0, 1, 2, \dots$.

The theorem is easy to prove by the well known result (Akhiezer [3] P.74) that states that any polynomial $R_n(x)$ of degree n which is non-negative and does not vanish identically on [0, 1] can be represented in the form

$$R_n(x) = x[A_m(x)]^2 + (1 - x)\,[B_m(x)]^2 ,$$

if $n = 2m + 1$ is odd, and in the form

$$R_n(x) = [C_m(x)]^2 + x(1 - x)\,[D_{m-1}(x)]^2$$

if $n = 2m$ is even. Here $A_m(x)$, $B_m(x)$, $C_m(x)$ and $D_{m-1}(x)$ are real polynomials the degree of which are given by their suffixes.

We claim that L is positive-definite on [0, 1] if and only if L, $\{x\}^*L$, $\{1-x\}^*L$ and $\{x(1-x)\}^*L$ are positive-definite.

Obviously, L is positive-definite on [0, 1] implies L is positive-definite. Note that any polynomial $\pi(x)$ which is non-negative and does not vanish identically on **R** implies $x\pi(x)$, $(1 - x)\pi(x)$ and $x(1 - x)\pi(x)$ are non-negative on [0, 1] and does not vanish identically. By the definitions of $\{x\}^*L$, $\{1-x\}^*L$, $\{x(1-x)\}^*L$ and the fact that L is positive-definite on [0, 1] , we have $\{x\}^*L$, $\{1-x\}^*L$ and $\{x(1-x)\}^*L$ are positive-definite.

Conversely, let L, $\{x\}^*L$, $\{1-x\}^*L$ and $\{x(1-x)\}^*L$ be positive-definite. For any polynomial $\pi_n(x)$ of degree n which is non-negative on [0, 1] and does not vanish identically.

(i) If n is even, then $\pi_n(x) = (C_m(x))^2 + x(1 - x)\,(D_{m-1}(x))^2 \quad n = 2m$.

By the positivity of L and $\{x(1 - x)\}^*L$, we have

$$L[\pi_n(x)] = L[(C_m(x))^2] + L[x(1 - x)\,(D_{m-1}(x))^2]$$

$$= L[(C_m(x))^2] + \{x(1 - x)\}^*L[(D_{m-1}(x))^2] > 0 \ .$$

(ii) If n is odd, then $\pi_n(x) = x(A_m(x))^2 + (1 - x)\,(B_m(x))^2 \quad n = 2m + 1$.

By the positivity of $\{x\}^*L$ and $\{1 - x\}^*L$, we have

$$L[\pi_n(x)] = L[x(A_m(x))^2] + L[(1 - x)\,(B_m(x))^2]$$

$$= \{x\}^*L[(A_m(x))^2] + \{1 - x\}^*L[(B_m(x))^2] > 0 \ .$$

Therefore L is positive-definite on [0, 1].

Now, it is easy to see that Theorem 3.3.1 follows by using Theorem 1.2.8 .

Q.E.D.

We have already noted that L being positive-definite guarantees that the corresponding representative ψ has an infinite spectrum. If we change all the " > " to " ≥ " in Theorem 1.6.1, Theorem 3.2.4 and Theorem 3.3.1, we still can find a representative ψ for the corresponding moment problem. In this case, the spectrum of ψ may only have finite number of points and the corresponding moment functional L is non-negative-definite.

For the Hausdorff moment problem, we will give another characterization which was given by Hausdorff himself. It is similar to what Akhiezer does in ([3] P.74).

Theorem 3.3.2 (Hausdorff[1]) Given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \ldots\}$, there exists a distribution function ψ whose spectrum (not necessary infinite) is contained in [0, 1] such that

$$\mu_n = \int_0^1 x^n \, d\,\psi(x) \qquad n = 0, 1, 2, \dots ,$$

if and only if the inequalities

$$\Delta^m \mu_k \equiv \sum_{i=0}^{m} (-1)^i \binom{m}{i} \mu_{i+k} \geq 0 \quad m, k = 0, 1, 2, \dots . \qquad (3.3.3)$$

Proof: First, we note that (3.3.3) can be represented in the form

$$L[x^k(1 - x)^m] \geq 0 \qquad m,k = 0, 1, 2, \dots . \qquad (3.3.4)$$

The fact that the condition is necessary is trivial since

$$x^k(1 - x)^m \geq 0 \qquad (0 \leq x \leq 1).$$

We now study the proof of sufficiency. Its essential part is the derivation of the identity

$$\sum_{k=0}^{N} \binom{N}{k} R\left(\frac{k}{N}\right) x^k (1 - x)^{N-k} = R(x) + \sum_{i=1}^{n-1} \frac{E_i(x)}{N^i}, \qquad (3.3.5)$$

where $R(x)$ is an arbitrary polynomial of degree n, the $E_i(x)$ are polynomials of degrees $\leq n$ in x which are independent of N, and $N \geq n$. The left hand side of identity (3.3.5) is the so-called Bernstein polynomial for $R(x)$.

Let us assume for the moment that (3.3.5) has been established.

We must prove that the inequality (3.3.4) implies the non-negative property of the functional L. Assume that $R(x)$ is an arbitrary polynomial of degree n which is non-negative on [0, 1] and does not vanish identically. Taking $N \geq n$ and applying the functional L to both sides of (3.3.5) we find that

$$L[\,R(x)\,] = \sum_{k=0}^{N} \binom{N}{k} R\left(\frac{k}{N}\right) L[x^k(1 - x)^{N-k}]$$

$$- \sum_{i=1}^{n-1} \frac{1}{N^i} L[\,E_i(x)\,] \geq - \sum_{i=1}^{n-1} \frac{1}{N^i} L[\,E_i(x)\,].$$

Hence, using the limiting process $N \to \infty$ we find that

$$L\,[R(x)] \geq 0 \; .$$

This concludes the proof.

It remains for us to derive identity (3.3.5). To do this we let the operator (1/N) p (d/dp) act n times on the relation

$$\sum_{k=0}^{N} \binom{N}{k} p^k q^{N-k} = (p+q)^N ,$$

where $N \geq n$ and then put $p = x$, $q = 1 - x$ in the resulting formula. On the left we obtain the polynomial

$$\sum_{k=0}^{N} \binom{N}{k} \left(\frac{k}{N}\right)^n x^k (1-x)^{N-k} ,$$

while on the right there will be a certain polynomial in x, the degree of which is evidently n. Thus we have the equation

$$\sum_{k=0}^{N} \binom{N}{k} \left(\frac{k}{N}\right)^n x^k (1-x)^{N-k} = x^n + \varepsilon_N(x) ,$$

where $\varepsilon_N(x)$ is a polynomial of degree $\leq n$ in x, the coefficients of which are rational fractions of N with the denominator N^{n-1}. We have on the left hand side the Bernstein polynomial of degree N for the polynomial x^n. For $N \to \infty$ it tends to x^n uniformly in the interval [0, 1]. Therefore the remainder $\varepsilon_N(x)$ must have the form

$$\varepsilon_N(x) = \sum_{i=1}^{n-1} \frac{e_i(x)}{N^i} ,$$

where the $e_i(x)$ are polynomials of degree $\leq n$ in x which do not depend on N. Thus the required identity is proved for $R(x) = x^n$. But then it holds for any polynomial R(x) of degree $\leq n$.

Q.E.D.

Chapter 4

Some Characterizations for the Existence of a Solution to the Hausdorff and Complemented Hausdorff Moment Problems

4.1 Introduction

From the nature of the relationships between the sets on which the Hausdorff, Stieltjes and Hamburger moment problems are defined, (see (3.3.1), (3.2.1) and (1.6.1)), it is obvious that: (i) Every Hausdorff moment sequence is a Stieltjes moment sequence and (ii) Every Stieltjes moment sequence is a Hamburger moment sequence. It is interesting to note that statement (ii) follows directly from the two characterizations of the existence of a solution as given by (3.2.2) and (1.6.2) but statement (i) does not follow from the characterization (3.3.3) and (3.2.2). There is another characterization of Hausdorff moment sequence(It is given in Theorem 3.3.1) from which (i) does follow. It states that a necessary and sufficient condition for $\{\mu_n \mid n= 0, 1, 2, ...\}$ to be a Hausdorff moment sequence is

$$\Delta_n\{1\} > 0, \qquad \Delta_n\{x\} > 0, \qquad \Delta_n\{1 - x\} > 0$$

$$\text{and} \quad \Delta_n\{x(1 - x)\} > 0, \quad \text{for} \quad n = 0, 1, 2, \ldots \, . \qquad (4.1.1)$$

We will show that the characterization given by (4.1.1), of a Hausdorff moment sequence has superfluous inequalities.

In this chapter we will also discuss the moment problem on the disconnected set $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$, where $\alpha_2 < \alpha_3$. That is: given a real sequence $\{\mu_n \mid n = 0, 1, 2, ...\}$, find a distribution function ψ with an infinite spectrum contained in $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$, such that

$$\int_{-\infty}^{\alpha_2} x^n \, d\psi(x) + \int_{\alpha_3}^{\infty} x^n \, d\psi(x) = \mu_n, \quad n = 0, 1, 2, \dots . \quad (4.1.2)$$

We call this moment problem the *complemented Hausdorff moment problem* . $\{\mu_n \mid n = 0, 1, 2, \dots\}$ is a *complemented Hausdorff moment sequence* on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ if it satisfies (4.1.2).

The new results given in this chapter are:

(I) $\{\mu_n \mid n = 0, 1, 2, \dots\}$ is a Hausdorff moment sequence on [0, 1] if and only if

$$\Delta_n\{1\} > 0, \quad \Delta_n\{x\} > 0, \quad \Delta_n\{1 - x\} > 0, \quad \text{for} \quad n = 0, 1, 2, \dots ,$$

if and only if

$$\Delta_n\{1\} > 0, \qquad \Delta_n\{x(1 - x)\} > 0, \qquad \text{for} \quad n = 0, 1, 2, \dots .$$

(II) $\{\mu_n \mid n = 0, 1, 2, \dots\}$ is a complemented Hausdorff moment sequence on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ if and only if

$$\Delta_n\{1\} > 0 , \quad \Delta_n\{(x - \alpha_2)(x - \alpha_3)\} > 0, \quad \text{for} \quad n = 0, 1, 2, \dots .$$

4.2 Two New Characterizations for a Hausdorff Moment Sequence

In this section we will state and prove two new characterizations for the Hausdorff moment sequence.

Theorem 4.2.1 A sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \dots\}$ is a Hausdorff moment sequence on [0, 1] if and only if

$\Delta_n\{1\} > 0$, $\Delta_n\{x\} > 0$ and $\Delta_n\{1 - x\} > 0$, for $n = 0, 1, 2, \ldots$. (4.2.1)

Proof: First, by using a representation theorem (either Theorem 1.5.5 or Theorem 2.2.2), we know that $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a Hausdorff moment sequence on [0, 1] if and only if the corresponding moment functional L is positive-definite on [0, 1].

Next, note that L is positive-definite on [0, 1] implies: (i) L is positive-definite on $[0, \infty)$ and (ii) L is positive-definite on $(-\infty, 1]$.

From (i), with the aid of Theorem 3.1.3 by choosing $\kappa = 0$, we obtain L and $\{x\}^*L$ are positive-definite.

From (ii), with the aid of Theorem 3.1.4 by choosing $\omega = 1$, we obtain L and $\{1 - x\}^*L$ are positive-definite .

Conversely, assume L, $\{x\}^*L$ and $\{1 - x\}^*L$ are positive-definite. This implies: (i) L and $\{x\}^*L$ are positive-definite. By Theorem 3.1.3, we have $\xi_1 \geq 0$. (ii) L and $\{1 - x\}^*L$ are positive-definite. By Theorem 3.1.4, we obtain $\eta_1 \leq 1$. Therefore L is positive-definite on [0, 1].

Finally, by Theorem 1.2.8, we know that L, $\{x\}^*L$ and $\{1 - x\}^*L$ are positive-definite if and only if $\Delta_n\{1\} > 0$, $\Delta_n\{x\} > 0$ and $\Delta_n\{1 - x\} > 0$, for $n = 0, 1, 2, \ldots$.

Q.E.D.

Theorem 4.2.2 A sequence of real numbers $\{\mu_n \mid n =0, 1, 2, \ldots\}$ is a Hausdorff moment sequence on [0, 1] if and only if

$$\Delta_n\{1\} > 0 \quad \text{and} \quad \Delta_n\{x(1 - x)\} > 0 \quad \text{for } n = 0, 1, 2, \ldots . \quad (4.2.2)$$

Proof: By using a representation theorem we know that $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a Hausdorff moment sequence on [0, 1] if and only if the corresponding moment functional L is positive-definite on [0, 1]. Now we prove that L is positive-definite on [0, 1] if and only if L and $\{x(1 - x)\}^*L$ are positive-definite .

If L is positive-definite on [0, 1], then L is positive-definite. Moreover, for any polynomial $\pi(x)$ which is non-negative and does not vanish identically on **R**, we have that $x(1 - x)\pi(x)$ is non-negative on [0, 1] and does not vanish identically. Since

$$\{x(1 - x)\}^*L\,[\,\pi(x)\,] = L\,[\,x(1 - x)\pi(x)\,] > 0\ ,$$

therefore $\{x(1 - x)\}^*L$ is positive-definite.

Conversely, if L and $\{x(1 - x)\}^*L$ are positive-definite, then by Theorem 3.1.5 we have $0 \le \xi_1 < \eta_1 \le 1$. This implies that L is positive-definite on [0, 1].

Theorem 4.2.2 now follows from Theorem 1.2.8.

Q.E.D.

In a similar manner, we can prove a modified Hausdorff moment problem by allowing the spectrum to be contained in a finite interval $[\alpha_1, \alpha_2]$. We state the result as the following theorem:

Theorem 4.2.3 Given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, ...\}$, there exists a distribution function ψ with an infinite spectrum contained in $[\alpha_1, \alpha_2]$ such that

$$\mu_n = \int_{\alpha_1}^{\alpha_2} x^n\, d\psi(x) \qquad n = 0, 1, 2, \ldots , \quad (4.2.3)$$

if and only if

$$\Delta_n\{1\} > 0, \qquad \Delta_n\{x - \alpha_1\} > 0, \qquad \Delta_n\{\alpha_2 - x\} > 0,$$

for $\quad n = 0, 1, 2, \ldots$, (4.2.4)

if and only if

$$\Delta_n\{1\} > 0, \qquad \Delta_n\{(x - \alpha_1)(\alpha_2 - x)\} > 0,$$

for $\quad n = 0, 1, 2, \ldots$. (4.2.5)

4.3 A Characterization for a Complemented Hausdorff Moment Sequence

In order to find a characterization for a complemented Hausdorff moment sequence, we need the following two theorems:

Theorem 4.3.1 The complemented Hausdorff moment problem has a solution if and only if L is positive-definite on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$.

Proof: This is the special case of Theorem 2.2.1 when $m = 2$, $\alpha_1 = -\infty$ and $\alpha_4 = \infty$.

Q.E.D.

Theorem 4.3.2 L is positive-definite on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ if and only if L and $\{(x - \alpha_2)(x - \alpha_3)\}^*L$ are positive-definite.

Proof: Firstly, by Theorem 1.3.1, L is positive-definite on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ implies L is positive-definite .

Secondly, for any $\pi(x)$ which is non-negative and does not vanish identically on **R** implies $(x - \alpha_2)(x - \alpha_3)\pi(x)$ is non-negative and does not vanish identically on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$. Thus,

$$\{(x - \alpha_2)(x - \alpha_3)\}^*L[\pi(x)] = L[(x - \alpha_2)(x - \alpha_3)\pi(x)] > 0$$

Therefore, $\{(x-\alpha_2)(x-\alpha_3)\}^*L$ is positive-definite.

Conversely, if L and $\{(x-\alpha_2)(x-\alpha_3)\}^*L$ are positive-definite then, for any polynomial $\pi(x)$ which is non-negative on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ and does not vanish identically, we have by Theorem 2.3.1 $\pi(x) = A(x) + (x-\alpha_2)(x-\alpha_3)B(x)$ with $A(x) \geq 0$ and $B(x) \geq 0$. This gives us

$$L[\pi(x)] = L[A(x)] + L[(x-\alpha_2)(x-\alpha_3)B(x)]$$

$$= L[A(x)] + \{(x-\alpha_2)(x-\alpha_3)\}^*L[B(x)] > 0$$

Therefore, L is positive-definite on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$.

Q.E.D.

Theorem 4.3.3 A sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a complemented Hausdorff moment sequence on $(-\infty, \alpha_2]\cup[\alpha_3, \infty)$ if and only if

$$\Delta_n\{1\} > 0 , \qquad \Delta_n\{(x-\alpha_2)(x-\alpha_3)\} > 0 ,$$

$$\text{for} \qquad n = 0, 1, 2, \ldots . \qquad (4.3.1)$$

Proof: It is easy to see that Theorem 4.3.3 follows from Theorem 4.3.1, Theorem 4.3.2 and Theorem 1.2.8.

Q.E.D.

Chapter 5

Some Characterizations for the Existence of a Solution to the Moment Problem on a Finite Number of Intervals

5.1 Introduction

Let $\{\alpha_i \mid i = 1, ..., 2m\}$ be a finite sequence of real numbers such that $-\infty \le \alpha_1 < \alpha_2 < ... < \alpha_{2m} \le \infty$ and let

$$E = (\cup_{1 \le i \le m}[\alpha_{2i-1}, \alpha_{2i}]) \cap (-\infty, \infty). \qquad (5.1.1)$$

The *moment problem on E* can be stated in the following manner: given a sequence of real numbers $\{\mu_n \mid n = 0, 1, 2, ...\}$, find a distribution function ψ with an infinite spectrum contained in E such that

$$\int_E x^n \, d\psi(x) = \mu_n, \quad n = 0, 1, 2, ... \qquad (5.1.2)$$

$\{\mu_n \mid n = 0, 1, 2, ...\}$ is called a *moment sequence on E* if it satisfies (5.1.2).

In this chapter we give some characterizations for a moment sequence on E

The main results in this chapter are:

(I) Let E be defined by (5.1.1) with $-\infty < \alpha_1$ and $\alpha_{2m} < \infty$. $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0, \; \Delta_n\{(x - \alpha_1)(\alpha_{2m} - x)\} > 0,$$

$$\Delta_n\{\prod_{i=2}^{2m-1}(x - \alpha_i)\} > 0 \quad \text{and} \quad \Delta_n\{-\prod_{i=1}^{2m}(x - \alpha_i)\} > 0 \quad \text{for} \quad n = 0, 1, 2, ...$$

(II) Let E be defined by (5.1.1) and $\mathbf{A} = \{A_s(x) \mid A_s(x) = d_s\Pi_{i\in S}(x - \alpha_i)$, $|\alpha_i| < \infty$, $S \subseteq \{1, ..., 2m\}$, $d_s = \pm 1$, $A_s(x) \geq 0$ for $x \in E\}$. $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{A_s(x)\} > 0, \quad n = 0, 1, 2, \ldots \ , \text{ for all } A_s(x) \in \mathbf{A}.$$

(III) Let E be defined by (5.1.1) with $-\infty < \alpha_1$, $\alpha_{2m} = \infty$ and $\{\mu_n \mid n = 0, 1, 2, ...\}$ be a moment sequence which is associated with a determinate moment problem. $\{\mu_n \mid n = 0, 1, 2, ...\}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0,\ \Delta_n\{x - \alpha_1\} > 0 \text{ and } \Delta_n\{(x - \alpha_{2i})(x - \alpha_{2i+1})\} > 0 \ \ i = 1, ..., m-1.$$

5.2 Preliminaries

Now we prove some preliminaries which will be used in the next section.

Theorem 5.2.1 Let a and b be two real numbers such that $-\infty < a < b < \infty$ and let p(x) be any polynomial. Also let $\psi(x)$ and $\psi^*(x)$ be two distribution functions such that $\psi^*(a) = 0 = \psi(a)$. If for all non-negative integers n

$$\int_a^b x^n \, d\psi^*(x) = \int_a^b x^n p(x)\, d\psi(x),$$

then

$$\int_a^b g(x;t)\, d\psi^*(x) = \int_a^b g(x;t)p(x)\, d\psi(x),$$

where g(x;t) is the unit step function defined by (2.1.1) and $t \in \mathbf{R}$.

Proof: The result follows trivially for the case when $t \leq a$ or $t \geq b$ and for the case when $\psi^*(b) = 0 = \psi(b)$.

Now we discuss the case when $a < t < b$ and $\psi^*(b) \neq 0 \neq \psi(b)$. Let ε be a real number > 0 and let $K = \max_{a \le x \le b}|p(x)|$. Because $\psi(x)$ and $\psi^*(x)$ are both right continuous at t, there exists a δ_0 such that for all $0 < \delta < \delta_0$, $\max\{ \psi^*(t+\delta) - \psi^*(t), K(\psi(t+\delta) - \psi(t))\} < \varepsilon/4$.

Let us define the continuous real valued function $g(x; t, \delta)$ on the compact set [a, b] by

$$g(x; t, \delta) = \begin{cases} 1 & \text{if } a \le x \le t \\ 1 - \dfrac{x - t}{\delta} & \text{if } t < x \le t + \delta \\ 0 & \text{if } t + \delta < x \le b \end{cases} , \qquad (5.2.1)$$

where $0 < \delta < b - t$.

By Weierstrass' Apporoximation Theorem (Apostol [1] P.481) there exists a polynomial $\pi(x)$ such that for all x belonging to [a, b],

$$| g(x; t, \delta) - \pi(x) | < \frac{\varepsilon}{4\max\{ K\psi(b), \psi^*(b) \}} ,$$

Also,

$$\int_a^b (g(x; t, \delta) - g(x; t))\, d\psi^*(x) = \int_t^{t+\delta} (1 - \frac{x - t}{\delta})\, d\psi^*(x) \le \psi^*(t + \delta) - \psi^*(t)$$

and

$$\int_a^b (g(x; t, \delta) - g(x; t))p(x)\, d\psi(x) = \int_t^{t+\delta} (1 - \frac{x - t}{\delta})p(x)\, d\psi(x) \le K(\psi(t + \delta) - \psi(t)) .$$

Therefore,

$$\int_a^b g(x;t)\, d\psi^*(x) - \int_a^b g(x;t)p(x)\, d\psi(x)$$

$$= \int_a^b (g(x;t) - g(x;t,\delta))\, d\psi^*(x) + \int_a^b (g(x;t,\delta) - \pi(x))\, d\psi^*(x)$$

$$+ \int_a^b (\pi(x) - g(x;t,\delta)p(x)\, d\psi(x) + \int_a^b (g(x;t,\delta) - g(x;t))p(x)\, d\psi(x).$$

Thus,

$$\left| \int_a^b g(x;t)\, d\psi^*(x) - \int_a^b g(x;t)p(x)\, d\psi(x) \right| \leq$$

$$\left| \int_a^b (g(x;t) - g(x;t,\delta))\, d\psi^*(x) \right| + \left| \int_a^b (g(x;t,\delta) - \pi(x))\, d\psi^*(x) \right|$$

$$+ \left| \int_a^b (\pi(x) - g(x;t,\delta)p(x)\, d\psi(x) \right| + \left| \int_a^b (g(x;t,\delta) - g(x;t))p(x)\, d\psi(x) \right|$$

$$< (\psi^*(t+\delta) - \psi^*(t)) + \varepsilon/4 + \varepsilon/4 + K(\psi(t+\delta) - \psi(t))$$

$$\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon .$$

Therefore,

$$\int_a^b g(x;t)\, d\psi^*(x) = \int_a^b g(x;t)\, p(x)\, d\psi(x) .$$

Q.E.D.

We use this theorem to obtain an interesting result in spectral analysis.

Theorem 5.2.2 Let a, b, λ_1, λ_2 be four real numbers such that $a < \lambda_1 < \lambda_2 < b$ and let p(x) be a polynomial that is not identically equal to zero such that $p(x) < 0$ for $x \in (\lambda_1, \lambda_2)$. If L, $\{(x-a)(b-x)\}^*L$, $\{p(x)\}^*L$ and $\{(x-a)(b-x)p(x)\}^*L$

are all positive-definite, then there exists a distribution function $\psi(x)$ such that

$$L[x^n] = \int_a^b x^n \, d\psi(x)$$

and the spectrum of $\psi(x)$ does not have any points in (λ_1, λ_2).

Proof: Because both L and $\{(x-a)(b-x)\}^*L$ are positive-definite, by Theorem 4.2.3 there exists a distribution function $\psi(x)$ such that

$$L[x^n] = \int_a^b x^n \, d\psi(x)$$

and similarly because both $\{p(x)\}^*L$ and $\{(x-a)(b-x)p(x)\}^*L$ are positive-definite, there exists a distribution function $\psi^*(x)$ such that

$$\{p(x)\}^*L[x^n] = \int_a^b x^n \, d\psi^*(x) \ .$$

Combining those two integrals we obtain

$$\int_a^b x^n \, d\psi^*(x) = \{p(x)\}^*L[x^n] = L[p(x)x^n] = \int_a^b x^n p(x) \, d\psi(x) \, .$$

Now we apply Theorem 5.2.1 to obtain that for all $t \in [a, b]$,

$$\int_a^b g(x;t) \, d\psi^*(x) = \int_a^b g(x;t) \, p(x) \, d\psi(x) \ .$$

Therefore,

$$0 \le \psi^*(\lambda_2) - \psi^*(\lambda_1) = \int_{\lambda_1}^{\lambda_2} p(x) \, d\psi(x) \le 0 \, ,$$

which implies that

$$\psi^*(\lambda_2) - \psi^*(\lambda_1) = 0 = \int_{\lambda_1}^{\lambda_2} p(x)\, d\psi(x) .$$

But $p(x) < 0$ for $x \in (\lambda_1, \lambda_2)$, so $\psi(\lambda_2) - \psi(\lambda_1) = 0$, and the spectrum of $\psi(x)$ has no points in (λ_1, λ_2).

Q.E.D.

5.3 A Characterization for the Moment Sequence on a Finite Number of Compact Intervals

In order to prove the characterization for the moment sequence on E where E is defined by (5.1.1) with $-\infty < \alpha_1$ and $\alpha_{2m} < \infty$, we need the following theorem.

Theorem 5.3.1 Let E be defined by (5.1.1) with $-\infty < \alpha_1$ and $\alpha_{2m} < \infty$. The following two statements are equivalent.

(i) L is positive-definite on E.

(ii) L, $\{(x - \alpha_1)(\alpha_{2m} - x)\}^*L$, $\{\prod_{i=2}^{2m-1}(x - \alpha_i)\}^*L$ and $\{-\prod_{i=1}^{2m}(x - \alpha_i)\}^*L$

are all positive-definite.

Proof:

Let L, $\{(x - \alpha_1)(\alpha_{2m} - x)\}^*L$, $\{\prod_{i=2}^{2m-1}(x - \alpha_i)\}^*L$ and $\{-\prod_{i=1}^{2m}(x - \alpha_i)\}^*L$

all be positive-definite and apply Theorem 5.2.2 with $p(x) = (x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_{2m-1})$, $a = \alpha_1$ and $b = \alpha_{2m}$ to obtain a distribution function $\psi(x)$ such that

(i) $\sigma(\psi) \subseteq [a, b]$;

(ii) $\sigma(\psi) \cap (\alpha_{2i}, \alpha_{2i+1}) = \emptyset$, for i = 1, 2, ..., m-1 ;

(iii) $L[x^n] = \int_{\alpha_1}^{\alpha_{2m}} x^n \, d\psi(x)$, for $n = 0, 1, 2, \ldots$.

Thus, L is positive-definite on E.

Conversely, let L be positive-definite on E. By Theorem 1.3.1 we know that L is positive-definite. For any polynomial $\pi(x) \geq 0$ on E which does not vanish identically, we have that $(x - \alpha_1)(\alpha_{2m} - x)\pi(x) \geq 0$ on E. Therefore,

$$\{(x - \alpha_1)(\alpha_{2m} - x)\}^*L[\pi(x)] = L[(x - \alpha_1)(\alpha_{2m} - x)\pi(x)] > 0,$$

which implies that $\{(x - \alpha_1)(\alpha_{2m} - x)\}^*L$ is positive-definite.

In a similar manner, it can be shown that both $\{\prod_{i=2}^{2m-1}(x - \alpha_i)\}^*L$ and $\{-\prod_{i=1}^{2m}(x - \alpha_i)\}^*L$ are positive-definite.

Q.E.D.

Theorem 5.3.2 Let E be defined by (5.1.1) with $-\infty < \alpha_1$ and $\alpha_{2m} < \infty$. $\{ \mu_n \mid n = 0, 1, 2, \ldots \}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0, \quad \Delta_n\{(x - \alpha_1)(\alpha_{2m} - x)\} > 0, \quad \Delta_n\{\prod_{i=2}^{2m-1}(x - \alpha_i)\} > 0$$

$$\text{and} \quad \Delta_n\{-\prod_{i=1}^{2m}(x - \alpha_i)\} > 0, \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

Proof: Theorem 5.3.2 follows from Theorem 1.2.8, Theorem 2.2.1 and Theorem 5.3.1 .

Q.E.D.

5.4 Some Characterizations for the Moment Sequence on a Finite Number of Intervals

In this section we will prove the following two characterizations for the moment sequence on a finite number of intervals.

Theorem 5.4.1 Let E be defined by (5.1.1). Then $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a moment sequence on E if and only if

$$\Delta_n\{A_s(x)\} > 0, \quad n = 0, 1, 2, \ldots, \quad \text{for all } A_s(x) \in \mathbf{A}.$$

Proof: Firstly, by Theorem 2.2.1 we know that $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a moment sequence on E if and only if L is positive-definite on E, where $L[x^n] = \mu_n$.

Secondly, by Theorem 2.3.4, L is positive-definite on E if and only if $\{A_s(x)\}^*L$ is positive-definite for all $A_s(x) \in \mathbf{A}$.

Finally, Theorem 5.4.1 follows by using Theorem 1.2.8.

Q.E.D.

If $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is associated with a determinate moment problem, then we have the following result.

Theorem 5.4.2 Let E be defined by (5.1.1) with $-\infty < \alpha_1$, $\alpha_{2m} = \infty$ and $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ be a moment sequence which is associated with a determinate moment problem. Then $\{\mu_n \mid n = 0, 1, 2, \ldots\}$ is a moment sequence on E if and only if

$$\Delta_n\{1\} > 0, \ \Delta_n\{x - \alpha_1\} > 0 \text{ and } \Delta_n\{(x - \alpha_{2i})(x - \alpha_{2i+1})\} > 0 \quad i = 1, \ldots, m-1.$$

Proof: Since L and $\{x - \alpha_1\}^*L$ are positive-definite. By Theorem 3.2.5 there is a distribution function ψ, such that

$$\mu_n = L[x^n] = \int_{\alpha_1}^{\infty} x^n \, d\psi(x) \quad \text{and} \quad \sigma(\psi) \cap (-\infty, \alpha_1) = \varnothing \, .$$

Since L and $\{(x - \alpha_{2i})(x - \alpha_{2i+1})\}^{*}L$ are positive-definite $i = 1, 2, ..., m-1$.

By Theorem 4.3.3 there is a distribution function ψ_i such that

$$\mu_n = L[x^n] = \int_{-\infty}^{\alpha_{2i}} x^n \, d\psi_i(x) + \int_{\alpha_{2i+1}}^{\infty} x^n \, d\psi_i(x) \quad \text{and} \quad \sigma(\psi_i) \cap (\alpha_{2i}, \alpha_{2i+1}) = \varnothing, \; i = 1, 2, ..., m-1.$$

Since L is determinate, ψ and ψ_i are substantially equal. This implies that

$\sigma(\psi) \subseteq E$ and

$$\mu_n = L[x^n] = \int_{E} x^n \, d\psi(x) \quad .$$

Q.E.D.

There are similar results for the cases when $-\infty = \alpha_1$, $\alpha_{2m} < \infty$; $-\infty < \alpha_1$, $\alpha_{2m} < \infty$ and $-\infty = \alpha_1$, $\alpha_{2m} = \infty$.

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