A subspace of $\ell_2(X)$ without the approximation property

by

Christopher Chlebovec

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Master of Science (Mathematics)

Department of Mathematical Sciences
Lakehead University
August, 2010

To my family

and

Kristen

ACKNOWLEDGEMENTS

I wish to express my gratitude to my advisor Razvan Anisca. This thesis would not have been such a rewarding and enjoyable experience if not for his patience, guidance, and support.

I am also very appreciative of the financial support I have received from the Natural Sciences and Engineering Research Council of Canada (NSERC) and Lakehead University throughout the Master's program.

Last, but certainly not least, I am thankful to my family and Kristen for their unwavering love and encouragement.

TABLE OF CONTENTS

ABSTRACT

A subspace of $\ell_2(X)$ without the approximation property

by

Christopher Chlebovec

The aim of the thesis is to provide support to the following conjecture, which would provide an isomorphic characterization of a Hilbert space in terms of the approximation property: an infinite dimensional Banach space X is isomorphic to ℓ_2 if and only if every subspace of $\ell_2(X)$ has the approximation property.

We show that if X has cotype 2 and the sequence of Euclidean distances $\{d_n(X^*)\}_n$ of X^* satisfies $d_n(X^*) \geq \alpha(\log_2 n)^\beta$ for all $n \geq 1$ and some absolute constants $\alpha > 0$ and $\beta > 4$, then $\ell_2(X)$ contains a subspace without the approximation property.

CHAPTER I

Introduction

This is a thesis in Geometric Functional Analysis devoted to the study of structural properties of infinite dimensional Banach spaces.

Among all Banach spaces, the Hilbert space ℓ_2 is the "nicest" and most "regular". It has lots of symmetries and, in particular, all of its infinite dimensional subspaces are isomorphic to the entire space. This is not true even for such classical spaces as ℓ_p, L_p ($p \neq 2$), whose subspaces admit much more diversity.

In this thesis we concentrate on constructing (infinite dimensional) Banach spaces without the approximation property; in particular, such Banach spaces do not admit a Schauder basis, which is to say that they do not have an infinite dimensional coordinate system. We are looking for arguments which allow us to obtain these constructions inside Banach spaces from certain large classes of spaces. This would support the idea that such a phenomenon is not merely accidental, but it reflects a common behavior.

We discuss first, in Chapter III, the very important construction of Szankowski [9] from the late 70s, who obtained subspaces of ℓ_p ($p \neq 2$) without the approximation property. As observed in the same paper, his arguments turned out to be more general and can be easily adapted to obtain the following more general result: an infinite dimensional Banach space X contains a subspace without the approximation property, unless X is "very close" to a Hilbert space, which is to say that X has type $(2-\epsilon)$ and cotype $(2+\epsilon)$ for all $\epsilon > 0$.

The objective of the thesis is to provide support to the following conjecture, which would provide an isomorphic characterization of a Hilbert space in terms of the approximation property: an infinite dimensional Banach space X is isomorphic to ℓ_2 if and only if every subspace of $\ell_2(X)$ has the approximation property. It is known that the corresponding statement involving only subspaces of X is not true (see, for example, the discussion in Chapter V).

The proposed question is equivalent to finding subspaces without the approximation property in $\ell_2(X)$, for every X which is not isomorphic to ℓ_2. Due to Szankowski's result, one only has to consider the case when X is an infinite dimensional space which has type $(2-\epsilon)$ and cotype $(2+\epsilon)$ for all $\epsilon > 0$. We will investigate the problem under the additional assumption that there is a certain control on the sequence of Euclidean distances of X, $\{d_n(X)\}_n$. The type and cotype properties of X imply, in this case, estimates of the form $d_n(X) \leq c(\alpha)n^\alpha$ and $d_n(X^*) \leq c(\alpha)n^\alpha$ for all $\alpha > 0$ and $n \geq 1$ (see, for example, [7]). In the main result of the thesis, which is contained in Chapter IV, we show that we can obtain subspaces of $\ell_2(X)$ without the approximation property provided that the sequence $\{d_n(X^*)\}_n$ is bounded below by $\{C(\log_2 n)^\beta\}_n$, where C and β are absolute constants.

The result of the thesis and the discussion at the end of Chapter V suggest that it seems plausible to continue the investigation and obtain a positive answer to the following question: does $\ell_2(X)$ contain a subspace without the approximation property provided the sequence of Euclidean distances $\{d_n(X^*)\}_n$ is bounded below by $\{f(n)\}_n$ for some iterate f of log? This would "almost" prove the mentioned isomorphic characterization of a Hilbert space in terms of the approximation property.

CHAPTER II

Preliminaries

We will start with some basic definitions needed throughout. For all other notations and concepts not explained here, we refer the reader to [2] and [4].

Definition II.1. Let X be a vector space over $\mathbb{C}$. A norm on X is a real valued function $||\cdot|| : X \to \mathbb{R}$ that satisfies the following properties for all $x, y \in X$ and $\alpha \in \mathbb{C}$:

- $||x|| \geq 0$
- $||x|| = 0$ if and only if $x = 0$
- $||\alpha x|| = |\alpha| \cdot ||x||$
- $||x + y|| \leq ||x|| + ||y||$

A vector space X equipped with a norm $||\cdot||$ is called a normed vector space, or simply a normed space.

Definition II.2. A normed spaced X is called a Banach space if it is complete with respect to the metric induced by its norm.

Definition II.3. Let X be a Banach space. A sequence $\{e_i\}_{i=1}^{L}$ in X is said to be 1-unconditional if

$$\left|\left|\sum_{i=1}^{L} \theta_i e_i\right|\right| = \left|\left|\sum_{i=1}^{L} e_i\right|\right|$$

for all possible signs $\{\theta_i\}_{i=1}^{L}$.

Definition II.4. Let $\{X_i\}_{i\geq 1}$ be a sequence of Banach spaces and $1 \leq p < \infty$. The ℓ_p-direct sum of the sequence $\{X_i\}_{i\geq 1}$ is

$$(X_1 \oplus X_2 \oplus \cdots)_{\ell_p} = \left\{ x = (x_1, x_2, \ldots,) \in \prod_{i=1}^{\infty} X_i : ||x|| = \left(\sum_{i=1}^{\infty} ||x_i||_{X_i}^p \right)^{1/p} < \infty \right\}.$$

We will write $(\sum_{n\geq 1} \bigoplus X_n)_{\ell_p} = (X_1 \oplus X_2 \oplus \cdots)_{\ell_p}$ and if $X = X_i$ for all $i \geq 1$, $\ell_p(X) = (X \oplus X \oplus \cdots)_{\ell_p}$.

We start with some preliminary results:

Proposition II.5. *Let X and Y be Banach spaces. Let $\{e_i\}_{i=1}^{L}$ and $\{f_j\}_{j=1}^{K}$ be a sequence of 1-unconditional vectors in X and Y, respectively. Then, $\{(e_i, 0))\}_{i=1}^{L} \cup \{(0, f_j)\}_{j=1}^{K}$ is a 1-unconditional sequence in $X \oplus_2 Y$.*

Proof. In $X \oplus_2 Y$ we have,

$$\begin{aligned}
\left|\left| \sum_{i=1}^{L} \theta_i(e_i, 0) + \sum_{j=1}^{K} \eta_j(0, f_j) \right|\right|_{X\oplus_2 Y} &= \left|\left| \left(\sum_{i=1}^{L} \theta_i e_i, \sum_{j=1}^{K} \eta_j f_j \right) \right|\right|_{X\oplus_2 Y} \\
&= \left(\left|\left| \sum_{i=1}^{L} \theta_i e_i \right|\right|_X^2 + \left|\left| \sum_{j=1}^{K} \eta_j f_j \right|\right|_Y^2 \right)^{1/2} \\
&= \left(\left|\left| \sum_{i=1}^{L} e_i \right|\right|_X^2 + \left|\left| \sum_{j=1}^{K} f_j \right|\right|_Y^2 \right)^{1/2} \\
&= \left|\left| \left(\sum_{i=1}^{L} e_i, \sum_{j=1}^{K} f_j \right) \right|\right|_{X\oplus_2 Y} \\
&= \left|\left| \sum_{i=1}^{L} (e_i, 0) + \sum_{j=1}^{K} (0, f_j) \right|\right|_{X\oplus_2 Y}
\end{aligned}$$

for all possible signs $\{\theta_i\}_{i=1}^{L}$ and $\{\eta_j\}_{j=1}^{K}$. $\square$

Proposition II.6. *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\{X_i\}_{i\geq 1}$ be a sequence of Banach spaces. Then, the dual of $\left(\sum_{i\geq 1} \bigoplus X_i\right)_{\ell_p}$ is isometrically isomorphic to $\left(\sum_{i\geq 1} \bigoplus X_i^*\right)_{\ell_q}$.*

Proof. Let $X = \left(\sum_{i\geq 1} \bigoplus X_i\right)_{\ell_p}$ and $Y = \left(\sum_{i\geq 1} \bigoplus X_i^*\right)_{\ell_q}$. Let

$$X^* \xrightarrow{T} Y$$

$$f \longmapsto (x_i^*)_i$$

where $x_i^*(x_i) = f(0, \ldots, 0, x_i, 0, \ldots)$ for each $x_i \in X_i$. Clearly T is linear and each $x_i^* \in X_i^*$ since

$$|x_i^*(x_i)| = |f(0, \ldots, 0, x_i, 0, \ldots) \leq ||f|| \cdot ||(0, \ldots, 0, x_i, 0, \ldots)||_X = ||f|| \cdot ||x_i||_{X_i},$$

which implies that $||x_i^*||_{X_i^*} \leq ||f|| < \infty$. We will use below that for each $0 < \delta < 1$, there exists a $y_n \in X_n$ with $x_n^*(y_n) \geq \delta ||x_n^*||$ and $||y_n|| = 1$. Now, put $z_n = ||x_n^*||^{q-1} y_n$. Then, for each k,

$$\begin{aligned}
\delta \left(\sum_{n=1}^{k} ||x_n^*||_{X_n^*}^q \right) &= \sum_{n=1}^{k} ||x_n^*||_{X_n^*}^{q-1} \cdot \delta ||x_n^*||_{X_n^*} \\
&\leq \sum_{n=1}^{k} ||x_n^*||_{X_n^*}^{q-1} \cdot x_n^*(y_n) \\
&= f(z_1, z_2, \ldots, z_k, 0, 0, \ldots) \\
&\leq ||f|| \cdot ||(z_1, z_2, \ldots, z_k, 0, 0, \ldots)||_X \\
&= ||f|| \cdot \left(\sum_{n=1}^{k} ||z_n||_{X_n}^p \right)^{1/p}
\end{aligned}$$

$$= ||f|| \cdot \left(\sum_{n=1}^{k} ||x_n^*||_{X_n^*}^{(q-1)p} \right)^{1/p}.$$

Since $(q-1)p = q$ and $1 - \frac{1}{p} = \frac{1}{q}$ we get that $\delta \left(\sum_{n=1}^{k} ||x_n^*||^q \right)^{1/q} \leq ||f||$ holds for each k and $0 < \delta < 1$. Hence, $||(x_i^*)_i||_Y \leq ||f||_{X^*} < \infty$. Also, for $(x_i)_i \in X$

$$\begin{aligned}
|f(x_1, x_2, \ldots)| &= \left| \sum_{n=1}^{\infty} x_n^*(x_n) \right| \\
&\leq \sum_{n=1}^{\infty} |x_n^*(x_n)| \\
&\leq \sum_{n=1}^{\infty} ||x_n^*||_{X_n^*} \cdot ||x_n||_{X_n} \\
&\leq \left(\sum_{n=1}^{\infty} ||x_n^*||_{X_n^*}^q \right)^{1/q} \left(\sum_{n=1}^{\infty} ||x_n||_{X_n}^p \right)^{1/p}.
\end{aligned}$$

Therefore, $||f||_{X^*} \leq ||(x_i^*)_i||_Y$ so that $||f||_{X^*} = ||(x_i^*)_i||_Y$.

To prove surjectivity let $(y_1^*, y_2^*, \ldots) \in (X_1^* \oplus X_2^* \oplus \cdots)_{\ell_q}$ and define $h \in (X_1 \oplus X_2 \oplus \cdots)_{\ell_p}^*$ by $h(y_1, y_2, \ldots) = \sum_{n=1}^{\infty} y_n^*(y_n)$, where $(y_1, y_2, \ldots) \in (X_1 \oplus X_2 \oplus \cdots)_{\ell_p}$.We have that h is well defined and thus $T(h) = (y_i^*)_i$. □

CHAPTER III

A subspace of ℓ_p $(p \neq 2)$ without the approximation property

The purpose of this chapter is to present the argument of Szankowski from [9] in which he obtains subspaces of ℓ_p $(p \neq 2)$ without the approximation property.

Definition III.1. A Banach space X has the approximation property if, for every compact set K in X and $\epsilon > 0$, there is a finite rank operator T on X so that $\|Tx - x\| \leq \epsilon$ for every $x \in K$. A weaker property is obtained if we only require the operator T to be compact, in which case the space X is said to have the compact approximation property (*C.A.P.*).

Examples of Banach spaces with the approximation property include all spaces with a Schauder basis; a sequence $\{x_i\}_i$ in X forms a Schauder basis for X if every element $x \in X$ has a unique representation as an infinite series $x = \sum_i a_i x_i$, for some scalars $\{a_i\}_i$. In order to check that such Banach spaces have the approximation property, one can always verify the definition above for an operator T chosen from one of the finite dimensional natural projections $\{P_n\}_n$, defined as $P_n(x) = \sum_{i=1}^n a_i x_i$ for all $x = \sum_n a_i x_i$.

The following criterion of a Banach space not having the *C.A.P.* is a modification of Enflo's original [3].

Proposition III.2. *Let X be a Banach space. Assume that there are sequences $\{x_j\}_{j=1}^\infty$ and $\{x_j^*\}_{j=1}^\infty$ in X and X^* respectively, a sequence $\{F_n\}_{n=1}^\infty$ of finite subsets of X and an increasing sequence of integers $\{k_n\}_{n=1}^\infty$ so that the following hold:*

(i) $x_j^*(x_j) = 1$ *for every* j

(ii) $x_j^* \xrightarrow{w*} 0$, $\sup_j ||x_j|| < \infty$

(iii) $|\beta_n(T) - \beta_{n-1}(T)| \leq \sup\{||Tx|| : x \in F_n\}$ *for every* $T \in L(X, X)$ *and* $n \geq 1$, *where* $\beta_0(T) = 0$ *and for* $n \geq 1$,

$$\beta_n(T) = k_n^{-1} \sum_{j=1}^{k_n} x_j^*(Tx_j)$$

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *where* $\gamma_n = \sup\{||x|| : x \in F_n\}$.

Then X does not have the C.A.P.

Proof. Let $T \in L(X, X)$ and $\epsilon > 0$. By (iv) there exists an $s_0 \geq 1$ with $\sum_{n=s_0}^{\infty} \gamma_n < \frac{\epsilon}{||T||}$. Now, let $r > s \geq s_0$. So, using (iii) we obtain

$$\begin{aligned}
|\beta_r(T) - \beta_s(T)| &= \left| \sum_{n=s+1}^{r} (\beta_n(T) - \beta_{n-1}(T)) \right| \\
&\leq \sum_{n=s+1}^{r} |\beta_n(T) - \beta_{n-1}(T)| \\
&\leq ||T|| \cdot \sum_{n=s+1}^{r} \gamma_n \\
&\leq ||T|| \cdot \sum_{n=s+1}^{\infty} \gamma_n \\
&< \epsilon
\end{aligned}$$

Thus, $\{\beta_n(T)\}_{n\geq 1}$ is convergent for every $T \in L(X,X)$ and so $\beta(T) := \lim_{n\to\infty} \beta_n(T)$ defines a linear functional β on $L(X,X)$. Let $\{\eta_n\}_{n\geq 1}$ be a sequence of positive numbers with $\eta_n \to \infty$ such that $C = \sum_{n=1}^{\infty} \eta_n\gamma_n < \infty$ and set $K = \bigcup_{n=1}^{\infty}(\eta_n\gamma_n)^{-1}F_n \cup \{0\}$. Clearly, K is countable and every nonzero element in K is of the form $(\eta_n\gamma_n)^{-1}y_n$, where $y_n \in F_n$. Since $||(\eta_n\gamma_n)^{-1}y_n|| = |(\eta_n\gamma_n)^{-1}|\,||y_n|| \leq \frac{1}{\eta_n}$ and $\frac{1}{\eta_n} \to 0$ as $n \to \infty$ we see that K is just a sequence tending to 0. Hence, K is compact.

If $y \in F_k$ we have $y = (\eta_k\gamma_k)x$ for some $x \in K$. Thus,

$$\begin{aligned}
|\beta_n(T)| &= \left|\sum_{k=1}^{n}(\beta_k(T) - \beta_{k-1}(T))\right| \\
&\leq \sum_{k=1}^{n}|\beta_k(T) - \beta_{k-1}(T)| \\
&\leq \sum_{k=1}^{n}\sup\{||Ty|| : y \in F_k\} \\
&\leq \left(\sum_{k=1}^{n}\eta_k\gamma_k\right)\sup\{||Tx|| : x \in K\}
\end{aligned}$$

Therefore, we have $|\beta(T)| \leq C\sup\{||Tx|| : x \in K\}$. Clearly, if I is the identity operator on X then $\beta_n(I) = 1$ for $n \geq 1$ and so $\beta(I) = 1$. We will prove below that $\beta(T) = 0$ for every compact operator $T \in L(X,X)$. This will conclude the proof since it will imply that

$$\sup\{||Tx - x|| : x \in K\} \geq C^{-1}|\beta(I - T)| = C^{-1}$$

for every compact operator $T \in L(X,X)$.

Let T be compact and let $\delta > 0$. Since $\sup_j ||x_j|| < \infty$ we have that $\{Tx_j\}_{j=1}^{\infty}$ is compact and hence totally bounded. Thus, we can pick points $\{y_i\}_{i=1}^{m}$ so that $\{B_\delta(y_i)\}_{i=1}^{m}$ is a finite cover of $\{Tx_j\}_{j=1}^{\infty}$. Equivalently, we have points $\{y_i\}_{i=1}^{m}$ so that for every j there is an $i(j)$ with $||Tx_j - y_{i(j)}|| \leq \delta$. For $n \geq 1$ we get that

$$\beta_n(T) = k_n^{-1} \sum_{j=1}^{k_n} x_j^*(Tx_j - y_{i(j)}) + k_n^{-1} \sum_{j=1}^{k_n} x_j^*(y_{i(j)})$$

and thus

$$|\beta_n(T)| \leq \delta \sup_j ||x_j^*|| + \sum_{i=1}^{m} k_n^{-1} \sum_{j=1}^{k_n} |x_j^*(y_i)|$$

For each $i \in \{1, \ldots, m\}$, we notice that $k_n^{-1} \sum_{j=1}^{k_n} |x_j^*(y_i)|$ is a Cesàro mean and since $x_j^* \xrightarrow{w*} 0$ we get $|\beta(T)| \leq \delta \sup_j ||x_j^*||$. Since $\delta > 0$ is arbitrary we get $|\beta(T)| = 0$. □

The next lemma is a combinatorial result which plays an important role in Szankowski's argument. Before proceeding to Lemma III.3, we will introduce some notations. For $n = 1, 2, \ldots$, let $\sigma_n = \{2^n, 2^n + 1, \ldots, 2^{n+1} - 1\}$. For each integer $j \geq 8$, we define nine integers $\{g_k(j)\}_{k=1}^{9}$ as follows:

$g_k(4i + l) = 2i + k - 1,$	$i = 2, 3, 4, \ldots,$	$l = 0, 1, 2, 3,$	$k = 1, 2$
$g_k(4i + l) = 4i + (l + k - 2) \bmod 4$	$i = 2, 3, 4, \ldots,$	$l = 0, 1, 2, 3,$	$k = 3, 4, 5$
$g_k(4i + l) = 8i + k - 6,$	$i = 2, 3, 4, \ldots,$	$l = 0, 1,$	$k = 6, 7, 8, 9$
$g_k(4i + l) = 8i + k - 2,$	$i = 2, 3, 4, \ldots,$	$l = 2, 3,$	$k = 6, 7, 8, 9$

Note that $g_k(\sigma_n) \subset \sigma_{n-1}$ for $k = 1, 2$; $g_k(\sigma_n) \subset \sigma_n$ for $k = 3, 4, 5$; and, $g_k(\sigma_n) \subset \sigma_{n+1}$ for $k = 6, 7, 8, 9$.

Lemma III.3. *There exist partitions Δ_n and ∇_n of σ_n into disjoint sets and a sequence of integers $\{m_n\}_{n=1}^{\infty}$ with $m_n \geq 2^{n/8-2}$, $n = 2, 3, 4 \ldots$, so that*

(i) If $A \in \nabla_n$, then $m_n \leq |A| \leq 2m_n$.

(ii) If $A \in \nabla_n$ and $B \in \Delta_n$ then $|A \cap B| \leq 1$.

(iii) For every $A \in \nabla_n$ and $1 \leq k \leq 9$, g_k is an injective function on A.

(iv) For every $A \in \nabla_n$, there is an element B of Δ_{n-1}, Δ_n or Δ_{n+1} such that

$g_k(A) \subset B$ *for all* $n \geq 3$ *and* $1 \leq k \leq 9$.

Proof. For $n \geq 2$ and $l = 0, 1, 2, 3$ we will denote $\sigma_n^l = \{j \in \sigma_n : j \equiv l \bmod 4\}$. Define $\varphi_n^l : \sigma_n^0 \to \sigma_n^l$ by $\varphi_n^l(j) = j + l$ and for $r = 0, 1$ define $\psi_{n,r} : \sigma_n^0 \to \sigma_{n+1}^0$ by $\psi_{n,r}(j) = 2j+4r$. The above maps are injective and $\varphi_n^l(\sigma_n^0) = \sigma_n^l$ and $\psi_{n,0}(\sigma_n^0) \cup \psi_{n,1}(\sigma_n^0) = \sigma_{n+1}^0$. Since $\{\sigma_n^l\}_{l=0}^3$ partitions σ_n and $\{\psi_{n,r}(\sigma_n^0)\}_{r=0}^1$ partitions σ_{n+1}^0, the maps have disjoint ranges.

For $n \geq 2$ we will represent σ_n^0 as $\sigma_n^0 = C_n \times D_n$, where

$$|D_{2m}| = |D_{2m+1}| = |C_{2m-1}| = |C_{2m}| = 2^{m-1}, \quad m = 1, 2, \ldots$$

so that for $n \geq 3$ we have the following:

- For each $c \in C_n$, there is an $r = 0, 1$ and a $d \in D_{n-1}$ with $\psi_{n-1,r}(C_{n-1} \times \{d\}) = \{c\} \times D_n$
- For each $c \in C_n$, there is a $d \in D_{n+1}$ with $\psi_{n,0}(\{c\} \times D_n) \cup \psi_{n,1}(\{c\} \times D_n) = C_{n+1} \times \{d\}$

Indeed, we will proceed inductively. Let $C_n = \{\overline{c_1}, \ldots, \overline{c}_{|C_n|}\}$, $D_n = \{\overline{d_1}, \ldots, \overline{d}_{|D_n|}\}$ such that $\sigma_n^0 = C_n \times D_n$ and by this we mean that there exists a bijection $F_n : C_n \times D_n \to \sigma_n^0$. Let $C_{n+1} = \{c_1, c_2, \ldots, c_{2|D_n|}\}$ and $D_{n+1} = \{d_1, d_2, \ldots, d_{|C_n|}\}$ be arbitrary sets of the prescribed cardinality. We will denote $C_{n+1}^{(1)} = \{c_1, c_2, \ldots, c_{|D_n|}\}$ and $C_{n+1}^{(2)} = \{c_{|D_n|+1}, c_{|D_n|+2}, \ldots, c_{2|D_n|}\}$ so that $C_{n+1} = C_{n+1}^{(1)} \cup C_{n+1}^{(2)}$. In order to obtain the claim for σ_{n+1}^0 we first define $F_{n+1,i} : C_{n+1} \times \{d_i\} \to \sigma_{n+1}^0$ by

$$F_{n+1,i}(c_j, d_i) = \begin{cases} \psi_{n,0}(F_n(\overline{c}_i, \overline{d}_j)) & \text{if} \quad c_j \in C_{n+1}^{(1)} \\ \psi_{n,1}(F_n(\overline{c}_i, \overline{d}_{j-|D_n|})) & \text{if} \quad c_j \in C_{n+1}^{(2)} \end{cases}.$$

Then, if we let $F_{n+1} = \bigcup_{i=1}^{|C_n|} F_{n+1,i}$, we get a bijection $F_{n+1} : C_{n+1} \times D_{n+1} \to \sigma_{n+1}^0$, which gives the representation $\sigma_{n+1}^0 = C_{n+1} \times D_{n+1}$. It is easy to see that if $c_j \in C_{n+1}^{(1)}$, $\psi_{n,0}(C_n \times \{\overline{d}_j\}) = \{c_j\} \times D_{n+1}$ and if $c_j \in C_{n+1}^{(2)}$, $\psi_{n,1}(C_n \times \{\overline{d}_{j-|D_n|}\}) = \{c_j\} \times D_{n+1}$. Also, $\psi_{n,0}(\{\overline{c}_i\} \times D_n) \cup \psi_{n,1}(\{\overline{c}_i\} \times D_n) = C_{n+1} \times \{d_i\}$ so that the above conditions are satisfied.

Now, having σ_n^0 represented as $\sigma_n^0 = C_n \times D_n$, we will represent D_n further as $D_n = \prod_{l=0}^{3} D_n^l$ so that

$$|D_n^0| \leq |D_n^1| \leq |D_n^2| \leq |D_n^3| \leq 2|D_n^0|.$$

We are now ready to define our partitions.

$$\nabla_n = \left\{ \varphi_n^l(\{f\} \times D_n^l) : f \in C_n \times \prod_{i \neq l} D_n^i, l = 0,1,2,3 \right\}$$

$$\Delta_n = \left\{ \varphi_n^l \left(C_n \times \prod_{i \neq l} D_n^i \times \{d\} \right) : d \in D_n^l, l = 0,1,2,3 \right\}$$

Let $m_n = |D_n^0|$ and pick an arbitrary $A \in \nabla_n$, say $A = \varphi_n^l(\{f\} \times D_n^l)$ for some $l = 0,1,2,3$ and $f = f' \times \prod_{i \neq l} f_i$, where $f' \in C_n$ and $f_i \in D_n^i$. We are now ready to prove the four claims of Lemma III.3:

(i) Clearly, $|A| = |D_n^l|$ and so $m_n \leq |A| \leq 2m_n$.

(ii) Let $B \in \Delta_n$ be such that $B = \varphi_n^s \left(C_n \times \prod_{i \neq s} D_n^i \times \{d\} \right)$, where $d \in D_n^s$. If $l \neq s$ then $A \cap B = \emptyset$. Otherwise we have $A \cap B = \{\varphi_n^l(f \times d)\}$.

(iii) Any element in A can be written as $4m + l$, for some $m \in \sigma_n$. Then, for all $k = 1, \ldots, 9$, it is easy to see that $g_k(4i + l) = g_k(4j + l)$ implies $i = j$ and hence g_k is injective on A.

(iv) To show $g_k(A)$ is contained in an element of either Δ_{n-1},Δ_n or Δ_{n+1} we will consider three cases: $k = 1,2$; $k = 3,4,5$; and $k = 6,7,8,9$.

- $k = 1, 2$: We have $g_k(A) = \varphi^{\beta}_{n-1}\psi^{-1}_{n-1,r}(\{f\} \times D^l_n)$, where for $k = 1$: $r = 0, \beta = 0$ (or $r = 1, \beta = 2$) and for $k = 2$: $r = 0, \beta = 1$ (or $r = 1, \beta = 3$). Since $f' \in C_n$ there is a $\gamma = 0$ or 1 and an $e \in D_{n-1}$ such that $\psi_{n-1,\gamma}(C_{n-1} \times \{e\}) = \{f'\} \times D_n$. So, we will choose $r = \gamma$ and then the corresponding β. Thus, $\varphi^{\beta}_{n-1}\psi^{-1}_{n-1,r}(\{f\} \times D^l_n) \subset \varphi^{\beta}_{n-1}\psi^{-1}_{n-1,r}(\{f'\} \times D_n) = \varphi^{\beta}_{n-1}(C_{n-1} \times \{e\})$ for some $e = \prod_{i=0}^{3} e_i$. Since $C_{n-1} \times \{e\} \subset C_{n-1} \times \prod_{i \neq \beta} D^i_n \times \{e_\beta\}$ we get

$$g_k(A) \subset \varphi^{\beta}_{n-1}(C_{n-1} \times \{e\}) \subset \varphi^{\beta}_{n-1}\left(C_{n-1} \times \prod_{i \neq \beta} D^i_n \times \{e_\beta\}\right) \in \Delta_{n-1}.$$

- $k = 3, 4, 5$: We have $g_k(A) = \varphi^s_n(\{f\} \times D^l_n)$, where $s \equiv (l + k - 2) \bmod 4$. Since $s \neq l$ we can write $f = f' \times \prod_{i \neq l, i \neq s} f_i \times f_s$, where $f_s \in D^s_n$. Thus, $\{f\} \times D^l_n \subset C_n \times \prod_{i \neq s} D^i_n \times \{f_s\}$. Therefore,

$$g_k(A) = \varphi^s_n(\{f\} \times D^l_n) \subset \varphi^s_n\left(C_n \times \prod_{i \neq s} D^i_n \times \{f_s\}\right) \in \Delta_n.$$

- $k = 6, 7, 8, 9$: We have $g_k(A) = \varphi^{\beta}_{n+1}\psi_{n,r}(\{f\} \times D^l_n)$, where $r = 0$ if $l = 0, 1$, $r = 1$ if $l = 2, 3$ and $\beta = 0, 1, 2, 3$. Since $f' \in C_n$, there is an $e \in D_{n+1}$ so that $\psi_{n,0}(\{f'\} \times D_n) \cup \psi_{n,1}(\{f'\} \times D_n) = C_{n+1} \times \{e\}$, for some $e = \prod_{i=0}^{3} e_i$ with $e_i \in D^i_{n+1}$. Thus, $\psi_{n,r}(\{f'\} \times D_n) \subset C_{n+1} \times \{e\}$ and so

$$\begin{aligned} g_k(A) &\subset \varphi^{\beta}_{n+1}\psi_{n,r}(\{f'\} \times D_n) \subset \varphi^{\beta}_{n+1}(C_{n+1} \times \{e\}) \\ &\subset \varphi^{\beta}_{n+1}\left(C_{n+1} \times \prod_{i \neq \beta} D^i_{n+1} \times \{e_\beta\}\right) \in \Delta_{n+1}. \end{aligned}$$

$\square$

Remark. We notice that if we set

$$\nabla_n = \Delta_n = \{\varphi_n^l(\{f\} \times D_n^l) : f \in C_n \times \prod_{i \neq l} D_n^i, l = 0, 1, 2, 3\}$$

the above proof is easily modified to obtain the results of Lemma III.5.

Theorem III.4. *For every $1 \leq p < 2$ the space ℓ_p has a subspace without the C.A.P.*

Proof. Let Δ_n be a partition of σ_n as given in Lemma III.3 Let $1 \leq p < 2$ and X be the space of all sequences $t = (t_i)_{i=1}^{\infty}$ such that

$$||t|| = \left(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \left(\sum_{j \in B} |t_i|^2 \right)^{p/2} \right)^{1/p} < \infty.$$

Thus, $X = \left(\sum_{n \geq 2} \sum_{A \in \Delta_n} \bigoplus \ell_2^{|A|} \right)_{\ell_p}$ which we know is isomorphic to a subspace of ℓ_p. Let $\{e_i\}_{i=4}^{\infty}$ be the unit vector basis of X and $\{e_i^*\}_{i=4}^{\infty}$ the biorthogonal functionals in X^*(i.e. $e_i^*(e_j) = \delta_{ij}$). By Proposition II.6 we have that

$$\left\| \sum_{i=4}^{\infty} t_i e_i^* \right\|_{X^*} = \left(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \left(\sum_{j \in B} |t_i|^2 \right)^{q/2} \right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now define $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$ and $Z = \overline{\text{span}}\{z_i\}_{i=2}^{\infty}$ which is a closed subspace of X. Define $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$ and for $T \in L(Z, Z)$ we put

$$\beta_n(T) = 2^{-n} \sum_{i \in \sigma_n} z_i^*(T z_i) \qquad n = 1, 2, 3 \ldots.$$

Using Proposition III.2 we will prove that Z does not have the *C.A.P.* Clearly (i) holds and for (ii) take $z \in Z$, say $z = \sum_{j=4}^{\infty} \lambda_j e_j$, since Z is a closed subspace of X. Then, $z_i^*(z) = \sum_{j=4}^{\infty} \lambda_j z_i^*(e_j) = \frac{1}{2}(\lambda_{2i} - \lambda_{2i+1})$. Since $|\lambda_i| \to 0$ we have that $z_i^* \xrightarrow{w*} 0$. So we are left to show (iii) and (iv) hold. We notice $(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)(z_i) = 4$

and $(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)(z_j) = 0$ when $j \neq i$. Therefore,

$$z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)|_Z = \frac{1}{4}(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)|_Z.$$

Hence, for $n \geq 2$ and $T \in L(Z, Z)$,

$$\beta_n(T) - \beta_{n-1}(T) =$$

$$2^{-n-1} \sum_{i \in \sigma_n} (e_{2i}^* - e_{2i+1}^*) T(e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + \cdots + e_{4i+3})$$

$$- 2^{-n-1} \sum_{i \in \sigma_{n-1}} (e_{4i}^* + \cdots + e_{4i+3}^*) T(e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + \cdots + e_{4i+3})$$

$$= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} (e_{4i}^* - e_{4i+1}^*) T(e_{4i} - e_{4i+1} + e_{8i} + e_{8i+1} + \cdots + e_{8i+3}) \\ +(e_{4i+2}^* - e_{4i+3}^*) T(e_{4i+2} - e_{4i+3} + e_{8i+4} + e_{8i+5} + \cdots + e_{8i+7}) \end{cases}$$

$$- 2^{-n-1} \sum_{i \in \sigma_{n-1}} (e_{4i}^* + \cdots + e_{4i+3}^*) T(e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + \cdots + e_{4i+3})$$

$$= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} e_{4i}^* T(e_{4i} - e_{4i+1} + e_{8i} + \cdots + e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ +e_{4i+1}^* T(-e_{4i} + e_{4i+1} - e_{8i} - \cdots - e_{8i+3} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ +e_{4i+2}^* T(e_{4i+2} - e_{4i+3} + e_{8i+4} + \cdots + e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \\ +e_{4i+3}^* T(-e_{4i+2} + e_{4i+3} - e_{8i+4} - \cdots - e_{8i+7} - e_{2i} + e_{2i+1} - e_{4i} - \cdots - e_{4i+3}) \end{cases}$$

$$= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} e_{4i}^* T(-e_{2i} + e_{2i+1} - 2e_{4i+1} - e_{4i+2} - e_{4i+3} + e_{8i} + \cdots + e_{8i+3}) \\ +e_{4i+1}^* T(-e_{2i} + e_{2i+1} - 2e_{4i} - e_{4i+2} - e_{4i+3} - e_{8i} - \cdots - e_{8i+3}) \\ +e_{4i+2}^* T(-e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - 2e_{4i+3} + e_{8i+4} + \cdots + e_{8i+7}) \\ +e_{4i+3}^* T(-e_{2i} + e_{2i+1} - e_{4i} - e_{4i+1} - 2e_{4i+2} - e_{8i+4} - \cdots - e_{8i+7}) \end{cases}$$

$$= 2^{-n-1} \sum_{j\in\sigma_{n+1}} e_j^*(Ty_j),$$

where

$$\sum_{k=1}^{9} \lambda_{j,k} e_{g_k(j)} = y_j \in Z \qquad j = 8, 9, 10, \dots,$$

and for every j, $|\lambda_{j,k}| = 1$ for eight indices k and $|\lambda_{j,k}| = 2$ for the ninth k.

For every $A \in \nabla_{n+1}$ we can write

$$\sum_{j\in A} e_j^* T y_j = 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j\in A} \theta_j e_j^* \right) T \left(\sum_{j\in A} \theta_j y_j \right) \right],$$

where $\sum_{\theta}$ is the summation taken over all possible signs $\{\theta_j\}_{j\in A}$.
Hence, we have that

$$\begin{aligned} \beta_n(T) - \beta_{n-1}(T) &= 2^{-n-1} \sum_{A\in\nabla_{n+1}} \sum_{j\in A} e_j^* T y_j \\ &= 2^{-n-1} \sum_{A\in\nabla_{n+1}} 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j\in A} \theta_j e_j^* \right) T \left(\sum_{j\in A} \theta_j y_j \right) \right]. \end{aligned}$$

For every $A \in \nabla_{n+1}$ $(n \geq 2)$ and $\{\theta_j\}_{j\in A}$ we have, by (ii) of Lemma III.3, that

$$\begin{aligned} \left\| \sum_{j\in A} \theta_j e_j^* \right\|_{Z^*} &\leq \left\| \sum_{j\in A} \theta_j e_j^* \right\|_{X^*} \\ &= \left(\sum_{B\in\Delta_{n+1}} \left(\sum_{j\in B\cap A} |\theta_j|^2 \right)^{q/2} \right)^{1/q} \\ &= \left(\sum_{j\in A} |\theta_j|^q \right)^{1/q} \\ &= |A|^{1/q} \\ &\leq (2m_{n+1})^{1/q}, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Let $E_n = \left\{ \sum_{j\in A} \theta_j y_j : A \in \nabla_{n+1}, \theta_j = \pm 1 \right\}$. Then,

$$
\begin{aligned}
|\beta_n(T) - \beta_{n-1}(T)| &\leq 2^{-n-1} \sum_{A\in\nabla_{n+1}} (2m_{n+1})^{1/q} \sup\{||Tz|| : z \in E_n\} \\
&\leq 2^{-n-1}(2^{n+1}m_{n+1}^{-1})(2m_{n+1})^{1/q} \sup\{||Tz|| : z \in E_n\} \\
&\leq 2m_{n+1}^{-1}m_{n+1}^{1/q} \sup\{||Tz|| : z \in E_n\} \\
&= 2m_{n+1}^{-1/p} \sup\{||Tz|| : z \in E_n\}.
\end{aligned}
$$

If we put $F_n = 2m_{n+1}^{-1/p}E_n$ we see that (iii) holds. So, we are left to show that (iv) holds. Define a sequence $\{\alpha_{g_k(j)}\}_{j\in A}$ by $\alpha_{g_k(j)} = \theta_j$ and note that it is well-defined by (iii) of Lemma III.3. By (iv) of Lemma III.3, $g_k(A) \subset B$, where B is an element of ∇_n, ∇_{n+1} or ∇_{n+2}. Hence, for every $A \in \nabla_{n+1}$ ($n \geq 2$) and $\{\theta_j\}_{j\in A}$ and every $1 \leq k \leq 9$,

$$
\begin{aligned}
\left|\left| \sum_{j\in A} \theta_j e_{g_k(j)} \right|\right| &= \left|\left| \sum_{j\in g_k(A)} \alpha_j e_j \right|\right| \\
&= \left(\sum_{j\in B\cap g_k(A)} |\alpha_j|^2 \right)^{1/2} \\
&= \left(\sum_{j\in g_k(A)} |\alpha_j|^2 \right)^{1/2} \\
&= \left(\sum_{j\in A} |\theta_j|^2 \right)^{1/2} \\
&= |A|^{1/2} \\
&\leq (2m_{n+1})^{1/2}
\end{aligned}
$$

and hence,

$$\left|\left|\sum_{j\in A}\theta_j y_j\right|\right| = \left|\left|\sum_{k=1}^{9}\lambda_{j,k}\sum_{j\in A}\theta_j e_{g_k(j)}\right|\right|$$

$$\leq \sum_{k=1}^{9}|\lambda_{j,k}|\left|\left|\sum_{j\in A}\theta_j e_{g_k(j)}\right|\right|$$

$$\leq (8+2)(2m_{n+1})^{1/2}$$

$$\leq 15m_{n+1}^{1/2}.$$

For every $A \in \nabla_{n+1}$ and $1 \leq p < 2$, we get that

$$\sup\{||x|| : x \in F_n\} \leq 2m_{n+1}^{-1/p}15m_{n+1}^{1/2}$$

$$= 30m_{n+1}^{1/2-1/p}$$

$$\leq C2^{\ell n},$$

where $\ell < 0$ and C is some constant, since $m_{n+1} \geq 2^{n+1/8-2}$. Therefore, (iv) holds as desired. □

Lemma III.5. *There exist partitions Δ_n and ∇_n of σ_n into disjoint sets and a sequence of integers $\{m_n\}_{m=1}^{\infty}$ with $m_n \geq 2^{n/8-2}$, $n = 2, 3, 4\ldots$, so that*

(i) If $A \in \nabla_n$, then $m_n \leq |A| \leq 2m_n$.

(ii) For every $A \in \nabla_n$ there is an element $B \in \Delta_n$ with $A \subset B$.

(iii) For every $A \in \nabla_n$ and $1 \leq k \leq 9$, g_k is an injective function on A.

(iv) For every $A \in \nabla_n$, $k = 1, \ldots, 9$, and every $B \in \Delta_{n-1}$, Δ_n or Δ_{n+1}, $|B \cap g_k(A)| \leq 1$.

Theorem III.6. *If $2 < p \leq \infty$, the space ℓ_p has a subspace without the C.A.P.*

Proof. Let Δ_n be a partition of σ_n as given in the above lemma III.5. Let $2 < p \leq \infty$ and X be the space of all sequences $t = (t_i)_{i=1}^{\infty}$ such that

$$||t|| = \left(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \left(\sum_{j \in B} |t_i|^2 \right)^{p/2} \right)^{1/p} < \infty.$$

Thus, $X = \left(\sum_{n \geq 2} \sum_{A \in \Delta_n} \bigoplus \ell_2^{|A|} \right)_{\ell_p}$ which we know is isomorphic to a subspace of ℓ_p. Let $\{e_i\}_{i=4}^{\infty}$ be the unit vector basis of X and $\{e_i^*\}_{i=4}^{\infty}$ the biorthogonal functionals in X^*(i.e. $e_i^*(e_j) = \delta_{ij}$). By Proposition II.6 we have that

$$\left|\left| \sum_{i=4}^{\infty} t_i e_i^* \right|\right|_{X^*} = \left(\sum_{n=2}^{\infty} \sum_{B \in \Delta_n} \left(\sum_{j \in B} |t_i|^2 \right)^{q/2} \right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Now define $z_i = e_{2i} - e_{2i+1} + e_{4i} + e_{4i+1} + e_{4i+2} + e_{4i+3}$ and $Z = \overline{\text{span}}\{z_i\}_{i=2}^{\infty}$ which is a closed subspace of X. Define $z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)$ and for $T \in L(Z, Z)$ we put

$$\beta_n(T) = 2^{-n} \sum_{i \in \sigma_n} z_i^*(T z_i) \qquad n = 1, 2, 3 \ldots.$$

Using Proposition III.2 we will prove that Z does not have the *C.A.P.* Clearly (i) holds and for (ii) take $z \in Z$, say $z = \sum_{j=4}^{\infty} \lambda_j e_j$, since Z is a closed subspace of X. Then, $z_i^*(z) = \sum_{j=4}^{\infty} \lambda_j z_i^*(e_j) = \frac{1}{2}(\lambda_{2i} - \lambda_{2i+1})$. Since $|\lambda_i| \to 0$ we have that $z_i^* \xrightarrow{w*} 0$. So we are left to show (iii) and (iv) hold. We notice $(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)(z_i) = 4$ and $(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)(z_j) = 0$ when $j \neq i$. Therefore,

$$z_i^* = \frac{1}{2}(e_{2i}^* - e_{2i+1}^*)|_Z = \frac{1}{4}(e_{4i}^* + e_{4i+1}^* + e_{4i+2}^* + e_{4i+3}^*)|_Z.$$

As seen in Theorem III.4, for $n \geq 2$ and $T \in L(Z, Z)$ we get

$$\beta_n(T) - \beta_{n-1}(T) = 2^{-n-1} \sum_{j \in \sigma_{n+1}} e_j^*(Ty_j),$$

where

$$\sum_{k=1}^{9} \lambda_{j,k} e_{f_k(j)} = y_j \in Z \qquad j = 8, 9, 10, \ldots,$$

and for every j, $|\lambda_{j,k}| = 1$ for eight indices k and $|\lambda_{j,k}| = 2$ for the ninth k.

For every $A \in \nabla_{n+1}$ we can write

$$\sum_{j \in A} e_j^* T y_j = 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j \in A} \theta_j e_j^* \right) T \left(\sum_{j \in A} \theta_j y_j \right) \right],$$

where $\sum_{\theta}$ is the summation taken over all possible signs $\{\theta_j\}_{j \in A}$. Hence, we have that

$$\begin{aligned} \beta_n(T) - \beta_{n-1}(T) &= 2^{-n-1} \sum_{A \in \nabla_{n+1}} \sum_{j \in A} e_j^* T y_j \\ &= 2^{-n-1} \sum_{A \in \nabla_{n+1}} 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j \in A} \theta_j e_j^* \right) T \left(\sum_{j \in A} \theta_j y_j \right) \right]. \end{aligned}$$

For every $A \in \nabla_{n+1}$ ($n \geq 2$) and $\{\theta_j\}_{j \in A}$ we have that

$$\begin{aligned} \left\| \sum_{j \in A} \theta_j e_j^* \right\|_{Z^*} &\leq \left\| \sum_{j \in A} \theta_j e_j^* \right\|_{X^*} \\ &= \left[\left(\sum_{j \in A} |\theta_j|^2 \right)^{q/2} \right]^{1/q} \\ &= |A|^{1/2} \\ &\leq (2m_{n+1})^{1/2}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Let $E_n = \left\{ \sum_{j \in A} \theta_j y_j : A \in \nabla_{n+1}, \theta_j = \pm 1 \right\}$. Then,

$$
\begin{aligned}
|\beta_n(T) - \beta_{n-1}(T)| &\leq 2^{-n-1} \sum_{A \in \nabla_{n+1}} (2m_{n+1})^{1/2} \sup\{||Tz|| : z \in E_n\} \\
&\leq 2^{-n-1}(2^{n+1} m_{n+1}^{-1})(2m_{n+1})^{1/2} \sup\{||Tz|| : z \in E_n\} \\
&\leq 2m_{n+1}^{-1} m_{n+1}^{1/2} \sup\{||Tz|| : z \in E_n\} \\
&= 2m_{n+1}^{-1/2} \sup\{||Tz|| : z \in E_n\}.
\end{aligned}
$$

If we put $F_n = 2m_{n+1}^{-1/2} E_n$ we see that (iii) holds. So, we are left to show that (iv) holds. Let $A \in \nabla_{n+1}$ $(n \geq 2)$ and $\{\theta_j\}_{j \in A}$ and every $1 \leq k \leq 9$. Define a sequence $\{\alpha_{g_k(j)}\}_{j \in A}$ by $\alpha_{g_k(j)} = \theta_j$ and note that it is well-defined by (iii) of Lemma III.5. Using the fact that $g_k(A) \subset \sigma_m$, where $m = n, n+1$, or $n+2$ (see the statement preceding Lemma III.3) as well as (iii) and (iv) of Lemma III.5, we have

$$
\begin{aligned}
\left\| \sum_{j \in A} \theta_j e_{g_k(j)} \right\| &= \left\| \sum_{j \in g_k(A)} \alpha_j e_j \right\| \\
&= \left(\sum_{B \in \Delta_m} \left(\sum_{j \in B \cap g_k(A)} |\alpha_j|^2 \right)^{p/2} \right)^{1/p} \\
&= \left(\sum_{j \in g_k(A)} |\alpha_j|^p \right)^{1/p} \\
&= \left(\sum_{j \in A} |\theta_j|^p \right)^{1/p} \\
&= |A|^{1/p} \\
&\leq (2m_{n+1})^{1/p}
\end{aligned}
$$

and hence,

$$
\begin{aligned}
\left|\left|\sum_{j\in A}\theta_j y_j\right|\right| &= \left|\left|\sum_{k=1}^{9}\lambda_{j,k}\sum_{j\in A}\theta_j e_{g_k(j)}\right|\right| \\
&\leq \sum_{k=1}^{9}|\lambda_{j,k}|\left|\left|\sum_{j\in A}\theta_j e_{g_k(j)}\right|\right| \\
&\leq (8+2)(2m_{n+1})^{1/p} \\
&\leq 15m_{n+1}^{1/p}.
\end{aligned}
$$

For every $A \in \nabla_{n+1}$ and $2 < p \leq \infty$ we get that

$$
\begin{aligned}
\sup\{||x|| : x \in F_n\} &\leq 2m_{n+1}^{-1/2}15m_{n+1}^{1/p} \\
&= 30m_{n+1}^{1/p-1/2} \\
&\leq C2^{\ell n},
\end{aligned}
$$

where $\ell < 0$ and C is some constant, since $m_{n+1} \geq 2^{n+1/8-2}$. Therefore, (iv) holds as desired. □

Remark. It was observed by Szankowski, in the same paper [9], that the above arguments can be easily adapted to obtain the following more general result: if X is an infinite dimensional Banach space, which contains ℓ_p^n's uniformly for some $p \neq 2$ then X contains a subspace without the *C.A.P.* Combining this result with the Maurey-Pisier theorem one obtains the following:

Theorem III.7. *Let X be an infinite dimensional Banach space. Then, X contains a subspace without the C.A.P. provided one of the following conditions hold:*

$$p_0^{(X)} = \sup\{p : X \textit{ has type } p\} < 2$$

or

$$q_0^{(X)} = \inf\{q : X \textit{ has cotype } q\} > 2.$$

Definition III.8. Given $1 \le p \le 2$, we say that X is of type p if there exists a $C_p > 0$ such that for all $x_1, \dots, x_n \in X$,

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \le C_p \left(\sum_{i=1}^n ||x_i||^p \right)^{1/p},$$

where r_i are the Rademacher functions, which are defined as $r_i(t) = \operatorname{sgn} \sin(2^i \pi t)$, $t \in [0,1]$. If X is of type p, we let $T_p(X)$ be the smallest such C_p that satisfies the above inequality for all $x_1, \dots, x_n \in X$ and all n.

Similarly, X is of cotype $q \ge 2$ if there exists a $C_q > 0$ such that for all $x_1, \dots, x_n \in X$,

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{1/2} \ge \frac{1}{C_q} \left(\sum_{i=1}^n ||x_i||^q \right)^{1/q}.$$

If X is of cotype q we let $C_q(X)$ to be the largest such C_q that satisfies the above inequality for all $x_1, \dots, x_n \in X$ and all n.

CHAPTER IV

A subspace of $\ell_2(X)$ without the approximation property

The purpose of this chapter is to provide sufficient conditions which imply that $\ell_2(X)$ contains a subspace without the approximation property. As discussed in Chapter III, one only has to consider infinite dimensional spaces X which are of type $(2-\epsilon)$ and cotype $(2+\epsilon)$ for all $\epsilon > 0$; otherwise, X itself will admit a subspace without the approximation property.

Definition IV.1. The Banach-Mazur distance $d(X,Y)$ between two Banach spaces X and Y is defined as

$$d(X,Y) = \inf\{||T|| \cdot ||T^{-1}|| : T \text{ is an isomorphism between } X \text{ and } Y\}.$$

We note that $d(X,Y) \geq 1$ and if X and Y are isometric then $d(X.Y) = 1$. If X and Y are not isomorphic we write $d(X,Y) = \infty$.

Definition IV.2. Let X be an infinite dimensional Banach space. The sequence of Euclidean distances $\{d_n(X)\}_n$ is defined as

$$d_n(X) = \sup\{d(Z, l_2^n) : Z \subset X, \dim Z = n\}.$$

It is clear that $d_n(X) \nearrow \infty$ as $n \to \infty$, for any infinite dimensional Banach space X which is not isomorphic to ℓ_2.

In the arguments below we will consider linear combinations with equal coefficients of certain 1-unconditional vectors. We require a behaviour in norm which we may not obtain in a given Banach space X, but can always get in $\ell_2(X)$. This is due to the following fact, which was originally formulated in terms of property (H) (see [8], Proposition 1.2): if Z is an n-dimensional Banach space there exists a universal constant $c > 0$ and $m \leq n$ normalized, 1-unconditional vectors $\{u_1, \ldots, u_m\} \subset \ell_2(Z)$, such that either

$$\left\| \sum_{j=1}^{m} u_j \right\|_{\ell_2(Z)} > cd(Z, l_2^n)^{1/4} m^{1/2}$$

or

$$\left\| \sum_{j=1}^{m} u_j \right\|_{\ell_2(Z)} < \frac{1}{cd(Z, l_2^n)^{1/4}} m^{1/2}.$$

Thus, given any infinite dimensional Banach space X, for all $n \geq 1$, there exists a universal constant $c > 0$ and $m \leq n$ normalized, 1-unconditional vectors $\{u_1, \ldots, u_m\} \subset \ell_2(X)$, such that either

$$\left\| \sum_{j=1}^{m} u_j \right\|_{\ell_2(X)} > cd_n(X)^{1/4} m^{1/2} \tag{4.1}$$

or

$$\left\| \sum_{j=1}^{m} u_j \right\|_{\ell_2(X)} < \frac{1}{cd_n(X)^{1/4}} m^{1/2}. \tag{4.2}$$

The existence of such vectors was essential for some other results which deal with the structure of subspaces of $\ell_2(X)$ (see for example [1] and [6]).

The following Lemma IV.3 and more specifically Corollary IV.5 describe the behaviour in norm of specific 1-unconditional normalized vectors necessary in constructing a subspace of $\ell_2(X)$ without the *C.A.P.*, as seen in the proof of Theorem IV.6.

Lemma IV.3. *Let X be a Banach space so that (4.2) holds for some N. Then, there exist 1-unconditional normalized vectors $\{z_i\}_{i=1}^N \subset \ell_2(X)$ such that*

$$\left\|\sum_{i=1}^N z_i\right\|_{\ell_2(X)} \leq \frac{\sqrt{2}}{c d_N(X)^{1/4}} N^{1/2}.$$

Proof. Let N be as described in condition (4.2). Then, there exists $M \leq N$ normalized, 1-unconditional vectors $\{y_i\}_{i=1}^M \subset \ell_2(X)$ such that

$$\left\|\sum_{i=1}^M y_i\right\|_{\ell_2(X)} \leq \frac{1}{c d_N(X)^{1/4}} M^{1/2}. \tag{4.3}$$

Suppose N is divisible by M, say $N = LM$ for some $L \geq 1$. We will now partition the set $\{1, \ldots, N\}$ into L sets of M elements and denote them by $A_1 = \{1, \ldots, M\}, A_2 = \{M+1, \ldots, 2M\}, \ldots, A_L = \{(L-1)M+1, \ldots, LM\}$. We can generate L sequences $\{y_i\}_{i\in A_1}, \ldots, \{y_i\}_{i\in A_L}$ of normalized, 1-unconditional vectors with

$$\left\|\sum_{i\in A_j} y_i\right\|_{\ell_2(X)} \leq \frac{1}{c d_N(X)^{1/4}} M^{1/2}$$

for every $1 \leq j \leq L$ by simply repeating and relabeling (4.3) $L-1$ times. For any $1 \leq j \leq L$ and $i \in A_j$, define $z_i \in \overbrace{\ell_2(X) \oplus_2 \cdots \oplus_2 \ell_2(X)}^{L \text{ copies}} \cong \ell_2(X)$ by $z_i = (0, \ldots, y_i, 0 \ldots, 0)$, where y_i is in the jth position. Then, by Proposition II.5 we have a set of normalized, 1-unconditional vectors $\{z_i\}_{i=1}^N$ in $\ell_2(X)$ with

$$\left\|\sum_{i=1}^N z_i\right\|_{\ell_2(X)} = \left\|\sum_{i\in A_1} z_i + \sum_{i\in A_2} z_i + \cdots + \sum_{i\in A_L} z_i\right\|_{\ell_2(X)}$$

$$= \left(\left\| \sum_{i \in A_1} y_i \right\|^2_{\ell_2(X)} + \cdots + \left\| \sum_{i \in A_L} y_i \right\|^2_{\ell_2(X)} \right)^{1/2}$$

$$\leq \left(L \cdot \frac{1}{c^2 d_N(X)^{1/2}} M \right)^{1/2}$$

$$= \frac{1}{c d_N(X)^{1/4}} N^{1/2}$$

Suppose on the other hand that N is not divisible by M and now take $L = \lfloor \frac{N}{M} \rfloor$. Then, $LM \leq N < (L+1)M$. Define $A_{L+1} = \{LM+1, \ldots, (L+1)M\}$ and by (4.2) pick again a sequence $\{y_i\}_{i \in A_{L+1}}$ which is 1-unconditional and normalized with

$$\left\| \sum_{i \in A_{L+1}} y_i \right\|_{\ell_2(X)} \leq \frac{1}{c d_N(X)^{1/4}} M^{1/2}$$

In this case we will partition $\{1, \ldots, N\}$ into the sets $A_1, \ldots, A_L, \{LM+1, \ldots, N\}$, where $\{LM+1, \ldots, N\} \subset A_{L+1}$. If we take $\{y_i\}_{i=LM+1}^N$, by 1-unconditionality we have

$$\left\| \sum_{i=LM+1}^{N} y_i \right\| \leq \left\| \sum_{i \in A_{L+1}} y_i \right\| \leq \frac{1}{c d_N(X)^{1/4}} M^{1/2}$$

We define z_i as before, but now for every $1 \leq j \leq L+1$ and $i \in A_j$ we have $z_i \in \overbrace{\ell_2(X) \oplus_2 \cdots \oplus_2 \ell_2(X)}^{L+1 \text{ copies}} \cong \ell_2(X)$. Finally,

$$\left\| \sum_{i=1}^{N} z_i \right\|_{\ell_2(X)} = \left\| \sum_{i \in A_1} z_i + \sum_{i \in A_2} z_i + \cdots + \sum_{i \in A_L} z_i + \sum_{i=LM+1}^{N} z_i \right\|_{\ell_2(X)}$$

$$= \left(\left\| \sum_{i \in A_1} y_i \right\|^2_{\ell_2(X)} + \cdots + \left\| \sum_{i \in A_L} y_i \right\|^2_{\ell_2(X)} + \left\| \sum_{i=LM+1}^{N} y_i \right\|^2_{\ell_2(X)} \right)^{1/2}$$

$$\leq \left((L+1) \cdot \frac{1}{c^2 d_N(X)^{1/2}} M \right)^{1/2}$$

$$\leq \frac{\sqrt{2}}{cd_N(X)^{1/4}} N^{1/2},$$

where the last inequality comes easily from the fact that $(L+1)M = LM + M \leq N + N = 2N$. $\square$

Lemma IV.4. *If X is a Banach space of type 2, then $\ell_2(X)$ is of type 2.*

Proof. For each $1 \leq i \leq n$, let $y_i = (x_1^i, \ldots, x_k^i, \ldots,) \in \ell_2(X)$. Then,

$$\begin{aligned}
\left(\int_0^1 \left|\left|\sum_{i=1}^n r_i(t) y_i\right|\right|_{\ell_2(X)}^2 dt\right)^{1/2} &= \left(\int_0^1 \left|\left|\sum_{i=1}^n r_i(t)(x_1^i, \ldots, x_k^i, \ldots)\right|\right|_{\ell_2(X)}^2 dt\right)^{1/2} \\
&= \left(\int_0^1 \left|\left|\left(\sum_{i=1}^n r_i(t) x_1^i, \ldots, \sum_{i=1}^n r_i(t) x_k^i, \ldots\right)\right|\right|_{\ell_2(X)}^2 dt\right)^{1/2} \\
&= \left(\int_0^1 \left|\left|\sum_{i=1}^n r_i(t) x_1^i\right|\right|_X^2 dt + \cdots + \int_0^1 \left|\left|\sum_{i=1}^n r_i(t) x_k^i\right|\right|_X^2 dt + \cdots\right)^{1/2} \\
&\leq \left(T_2(X)^2 \sum_{i=1}^n ||x_1^i||_X^2 + \cdots + T_2(X)^2 \sum_{i=1}^n ||x_k^i||_X^2 + \cdots\right)^{1/2} \\
&= T_2(X) \left(\sum_{i=1}^n (||x_1^i||_X^2 + \cdots + ||x_k^i||_X^2 + \cdots)\right)^{1/2} \\
&= T_2(X) \left(\sum_{i=1}^n ||(x_1^i, \ldots, x_k^i)||_{\ell_2(X)}^2\right)^{1/2} \\
&= T_2(X) \left(\sum_{i=1}^n ||y_i||_{\ell_2(X)}^2\right)^{1/2}.
\end{aligned}$$

$\square$

The next result shows that under the hypothesis of X having type 2, condition (4.2) must be satisfied for all N large enough.

Corollary IV.5. *Let X be a an infinite dimensional Banach space of type 2 which is not isomorphic to ℓ_2. Then, for all N large enough, there exist normalized, 1-unconditional vectors $\{z_i\}_{i=1}^N \subset \ell_2(X)$ such that*

$$\left\|\sum_{i=1}^{N} z_i\right\|_{\ell_2(X)} \leq \frac{\sqrt{2}}{cd_N(X)^{1/4}} N^{1/2}.$$

Proof. We claim that (4.1) holds for only finitely many N. Suppose on the contrary that (4.1) holds for infinitely many N. Then, we can find an infinite increasing sequence $\{k_n\}_{n\geq 1}$ and $m_n \leq k_n$ normalized 1-unconditional vectors $\{u_1, \dots, u_{m_n}\} \subset \ell_2(X)$ such that

$$\left\|\sum_{j=1}^{m_n} u_j\right\|_{\ell_2(X)} > cd_{k_n}(X)^{1/4} {m_n}^{1/2}.$$

Since X is of type 2, $\ell_2(X)$ is of type 2 by Lemma IV.4 and taking into account the 1-unconditionality we must have

$$\begin{aligned}
T_2(\ell_2(X)) m_n^{1/2} &= T_2(\ell_2(X)) \left(\sum_{j=1}^{m_n} ||u_j||^2\right)^{1/2} \\
&\geq \left(\int_0^1 \left\|\sum_{j=1}^{m_n} r_i(t) u_j\right\|_{\ell_2(X)}^2\right)^{1/2} \\
&= \left(\frac{1}{2^{m_n}} \sum_{\varepsilon_j=\pm 1} \left\|\sum_{j=1}^{m_n} \epsilon_j u_j\right\|_{\ell_2(X)}^2\right)^{1/2} \\
&= \left(\frac{1}{2^{m_n}} \sum_{\varepsilon_j=\pm 1} \left\|\sum_{j=1}^{m_n} u_j\right\|_{\ell_2(X)}^2\right)^{1/2} \\
&= \left\|\sum_{j=1}^{m_n} u_j\right\|_{\ell_2(X)} \\
&\geq cd_{k_n}(X)^{1/4} {m_n}^{1/2}
\end{aligned}$$

Therefore, $T_2(\ell_2(X)) \geq c d_{k_n}(X)^{1/4}$. Since X is not isomorphic to ℓ_2, $d_{k_n}(X) \to \infty$ as $n \to \infty$. This implies $T_2(\ell_2(X)) = \infty$, which is a contradiction. Hence, there is a finite number of values $N_1, \ldots, N_k$ for which (4.1) holds. So, we have that (4.2) holds if we take $N > \max\{N_1, \ldots, N_k\}$. Since (4.2) holds for all N large enough, by Lemma IV.3 we get the desired result. □

The proof of the main result of the thesis (Theorem IV.6) uses the same combinatorial result as Szankowski (see Lemma III.3 or III.5). We will use only one of the partitions, namely ∇_n, which basically satisfies the following property: for every $A \in \nabla_n$, $k = 1, \ldots, 9$, and every $A_0 \in \nabla_{n-1}$, ∇_n or ∇_{n+1}, $|A_0 \cap g_k(A)| \leq 1$ (see (ii) and (iv) of Lemma III.3 or III.5).

Theorem IV.6. *Let X be an infinite dimensional Banach space which is not isomorphic to ℓ_2. Assume that X has cotype 2 and $d_n(X^*) \geq \alpha(\log_2 n)^\beta$ for all $n \geq 1$ and some absolute constants $\alpha > 0$ and $\beta > 4$. Then, $\ell_2(X)$ has a subspace without the C.A.P.*

To put things into perspective, we should mention that an infinite dimensional Banach space X which has type $(2-\epsilon)$ and cotype $(2+\epsilon)$ for all $\epsilon > 0$, satisfies the following estimates for its sequences of Euclidean distances $\{d_n(X)\}_n$ and $\{d_n(X^*)\}_n$:

$$d_n(X) \leq c(\gamma)n^\gamma \text{ and } d_n(X^*) \leq c(\gamma)n^\gamma,$$

for all $\gamma > 0$ and $n \geq 1$ (see [7]).

Proof. We can assume that X does not contain ℓ_1^n 's uniformly, otherwise X itself will have a subspace without the *C.A.P.* by Szankowski's result. In such a case, X^* is of type 2 since X has cotype 2 (see [7]).

Let $m \geq 2$ be fixed and pick $A_0 \in \nabla_m$. Then, by Corolloary IV.5 there exist a set of normalized, 1-unconditional vectors $\{e_i^*\}_{i\in A_0}$ in $\ell_2(X^*)$ such that

$$\left|\left|\sum_{i\in A_0} e_i^*\right|\right|_{\ell_2(X^*)} \leq \frac{\sqrt{2}}{c(\log_2|A_0|)^\gamma}|A_0|^{1/2}$$

for some absolute constants $c > 0$ and $\gamma > 1$ (which do not depend on m or $|A_0|$).

Take $j \in A_0$ arbitrarily fixed and define $\widetilde{e}_j^{**} : \text{span}\{e_i^*\}_{i\in A_0} \longrightarrow \mathbb{R}$ by $\widetilde{e}_j^{**}(e_i^*) = \delta_{ij}$. By 1-unconditionality we have that $||a_k e_k^*|| \leq \left|\left|\sum_{i\in A_0} a_i e_i^*\right|\right|$ for every $k \in A_0$ and scalars $\{a_i\}_{i\in A_0}$. Thus,

$$\begin{aligned}
||\widetilde{e}_j^{**}|| &= \sup\left\{\left|\widetilde{e}_j^{**}\left(\sum_{i\in A_0} a_i e_i^*\right)\right| : \left|\left|\sum_{j\in A_0} a_j e_j^*\right|\right| = 1\right\} \\
&= \sup\left\{|a_j| : \left|\left|\sum_{j\in A_0} a_j e_j^*\right|\right| = 1\right\} \\
&= \sup\left\{||a_j e_j^*|| : \left|\left|\sum_{j\in A_0} a_j e_j^*\right|\right| = 1\right\} \\
&\leq 1
\end{aligned}$$

Since $\widetilde{e}_j^{**}(e_j) = 1$ we have that $||\widetilde{e}_j^{**}|| = 1$.

So, by the Hahn-Banach theorem there is an $e_j^{**} \in (\ell_2(X^*))^*$ with $e_j^{**}(e_i^*) = \delta_{ij}$ and $||e_j^{**}||_{(\ell_2(X^*))^*} = 1$. Since $(\ell_2(X^*))^* \cong \ell_2(X)^{**}$ by the principle of local reflexivity (see [10]) there exist elements $\{e_j\}_{j\in A_0} \subset \ell_2(X)$ satisfying $\frac{1}{2} \leq ||e_j|| \leq \frac{3}{2}$ and $e_i^*(e_j) = \delta_{ij}$ for each $i \in A_0$.

For every $A \in \nabla_m$ ($m \geq 2$), let $X_A = \text{span}\{e_i\}_{i\in A}$ and set $Y = \left(\sum_{m=2}^{\infty}\sum_{A\in\nabla_m} \bigoplus X_A\right)_{\ell_2}$. Clearly, Y is a subspace of $\ell_2(X)$ since X_A is a subspace of $\ell_2(X)$ and $\ell_2(\ell_2(X)) \cong \ell_2(X)$. Let $i \in A$ and define $f_i = (0, \ldots, 0, e_i, 0, \ldots)$, where $e_i \in X_A$. Then, the set of vectors $\{f_i\}_{i\geq 4}$ is a basis for Y since $\{e_i\}_{i\in A}$ is a basis for each corresponding X_A

$(A \in \nabla_m,\ m \geq 2)$. Therefore, any element of Y has the form $\sum t_i f_i$ with norm

$$\left\|\sum t_i f_i\right\|_Y = \left(\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \left\|\sum_{i \in A} t_i f_i\right\|_{\ell_2(X)}^2\right)^{1/2} = \left(\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \left\|\sum_{i \in A} t_i e_i\right\|_{\ell_2(X)}^2\right)^{1/2}.$$

We also notice that for each $i \in A$, $e_i^* \in \ell_2(X)^*$ since $\ell_2(X^*) \cong \ell_2(X)^*$ and thus $e_i^*\big|_{X_A} \in X_A^*$ with $e_i^*\big|_{X_A}(e_j) = \delta_{ij}$ for every $j \in A$ and $\left\|e_i^*\big|_{X_A}\right\| \leq 1$. Hence, $\left\{e_i^*\big|_{X_A}\right\}_{i \in A}$ is a basis for X_A^*. So, by Proposition II.6

$$Y^* = \left(\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \bigoplus X_A^*\right)_{\ell_2} \qquad \text{with } X_A^* = \operatorname{span}\left\{e_i^*\big|_{X_A}\right\}_{i \in A}.$$

Now define for $A \in \nabla_m$ and $i \in A$, $f_i^* = \left(\mathbf{0}, \ldots, \mathbf{0}, e_i^*\big|_{X_A}, \mathbf{0}, \ldots\right)$, where $e_i^*\big|_{X_A} \in X_A^*$. Thus, any element of Y^* is of the form $\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \sum_{i \in A} t_i f_i^*$ with norm

$$\left\|\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \sum_{i \in A} t_i f_i^*\right\|_{Y^*} = \left(\sum_{m=2}^{\infty} \sum_{A \in \nabla_m} \left\|\sum_{i \in A} t_i e_i^*\big|_{X_A}\right\|_{X_A^*}^2\right)^{1/2}.$$

Moreover, if $j \in A$ then $f_i^*(f_j) = e_i^*\big|_{X_A}(e_j) = \delta_{ij}$. If $j \notin A$, then $f_i^*(f_j) = e_i^*\big|_{X_A}(0) + \mathbf{0}(e_j) = 0$. Hence, $f_i^*(f_j) = \delta_{ij}$ for all $i, j \geq 4$.

We are now ready to construct our subspace of $\ell_2(X)$. As in the proof of Theorem III.4 define $z_i = f_{2i} - f_{2i+1} + f_{4i} + f_{4i+1} + f_{4i+2} + f_{4i+3}$ and $Z = \overline{\operatorname{span}}\{z_i\}_{i=2}^{\infty}$ which is a closed subspace of Y. Define $z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)$ and for $T \in L(Z, Z)$ we put

$$\beta_n(T) = 2^{-n} \sum_{i \in \sigma_n} z_i^*(T z_i) \qquad n = 1, 2, 3 \ldots.$$

Using Proposition III.2 we will prove that Z does not have the $C.A.P.$ Clearly (i) holds and for (ii) take $z \in Z$, say $z = \sum_{j=4}^{\infty} \lambda_j f_j$, since Z is a closed subspace of Y. Then,

$z_i^*(z) = \sum_{j=4}^{\infty} \lambda_j z_i^*(f_j) = \frac{1}{2}(\lambda_{2i} - \lambda_{2i+1})$. Since $|\lambda_i| \to 0$ we have that $z_i^* \xrightarrow{w*} 0$. So we are left to show that (iii) and (iv) hold. We notice $(f_{4i}^* + f_{4i+1}^* + f_{4i+2}^* + f_{4i+3}^*)(z_i) = 4$ and $(f_{4i}^* + f_{4i+1}^* + f_{4i+2}^* + f_{4i+3}^*)(z_j) = 0$ when $j \neq i$. Therefore,

$$z_i^* = \frac{1}{2}(f_{2i}^* - f_{2i+1}^*)|_Z = \frac{1}{4}(f_{4i}^* + f_{4i+1}^* + f_{4i+2}^* + f_{4i+3}^*)|_Z.$$

Hence, for $n \geq 2$ and $T \in L(Z, Z)$,

$$\beta_n(T) - \beta_{n-1}(T) =$$

$$2^{-n-1} \sum_{i \in \sigma_n} (f_{2i}^* - f_{2i+1}^*) T(f_{2i} - f_{2i+1} + f_{4i} + \cdots + f_{4i+3})$$

$$- 2^{-n-1} \sum_{i \in \sigma_{n-1}} (f_{4i}^* + \cdots + f_{4i+3}^*) T(f_{2i} - f_{2i+1} + f_{4i} + \cdots + f_{4i+3})$$

$$= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} (f_{4i}^* - f_{4i+1}^*) T(f_{4i} - f_{4i+1} + f_{8i} + \cdots + f_{8i+3}) \\ +(f_{4i+2}^* - f_{4i+3}^*) T(f_{4i+2} - f_{4i+3} + f_{8i+4} + \cdots + f_{8i+7}) \end{cases}$$

$$- 2^{-n-1} \sum_{i \in \sigma_{n-1}} (f_{4i}^* + \cdots + f_{4i+3}^*) T(f_{2i} - f_{2i+1} + f_{4i} + \cdots + f_{4i+3})$$

$$= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} f_{4i}^* T(f_{4i} - f_{4i+1} + f_{8i} + \cdots + f_{8i+3} - f_{2i} + f_{2i+1} - f_{4i} - \cdots - f_{4i+3}) \\ +f_{4i+1}^* T(-f_{4i} + f_{4i+1} - f_{8i} - \cdots - f_{8i+3} - f_{2i} + f_{2i+1} - f_{4i} - \cdots - f_{4i+3}) \\ +f_{4i+2}^* T(f_{4i+2} - f_{4i+3} + f_{8i+4} + \cdots + f_{8i+7} - f_{2i} + f_{2i+1} - f_{4i} - \cdots - f_{4i+3}) \\ +f_{4i+3}^* T(-f_{4i+2} + f_{4i+3} - f_{8i+4} - \cdots - f_{8i+7} - f_{2i} + f_{2i+1} - f_{4i} - \cdots - f_{4i+3}) \end{cases}$$

$$
= 2^{-n-1} \sum_{i \in \sigma_{n-1}} \begin{cases} f_{4i}^* T(-f_{2i} + f_{2i+1} - 2f_{4i+1} - f_{4i+2} - f_{4i+3} + f_{8i} + \cdots + f_{8i+3}) \\ +f_{4i+1}^* T(-f_{2i} + f_{2i+1} - 2f_{4i} - f_{4i+2} - f_{4i+3} - f_{8i} - \cdots - f_{8i+3}) \\ +f_{4i+2}^* T(-f_{2i} + f_{2i+1} - f_{4i} - f_{4i+1} - 2f_{4i+3} + f_{8i+4} + \cdots + f_{8i+7}) \\ +f_{4i+3}^* T(-f_{2i} + f_{2i+1} - f_{4i} - f_{4i+1} - 2f_{4i+2} - f_{8i+4} - \cdots - f_{8i+7}) \end{cases}
$$

$$
= 2^{-n-1} \sum_{j \in \sigma_{n+1}} f_j^*(Ty_j),
$$

where

$$
\sum_{k=1}^{9} \lambda_{j,k} f_{g_k(j)} = y_j \in Z \qquad j = 8, 9, 10, \ldots,
$$

and for every j, $|\lambda_{j,k}| = 1$ for eight indices k and $|\lambda_{j,k}| = 2$ for the ninth k.

We note that for every $A \in \nabla_{n+1}$ we can write

$$
\sum_{j \in A} f_j^* T y_j = 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j \in A} \theta_j f_j^* \right) T \left(\sum_{j \in A} \theta_j y_j \right) \right],
$$

where $\sum_{\theta}$ is the summation taken over all possible signs $\{\theta_j\}_{j \in A}$. Hence, we have that

$$
\begin{aligned}
\beta_n(T) - \beta_{n-1}(T) &= 2^{-n-1} \sum_{A \in \nabla_{n+1}} \sum_{j \in A} f_j^* T y_j \\
&= 2^{-n-1} \sum_{A \in \nabla_{n+1}} 2^{-|A|} \sum_{\theta} \left[\left(\sum_{j \in A} \theta_j f_j^* \right) T \left(\sum_{j \in A} \theta_j y_j \right) \right].
\end{aligned}
$$

For every $A \in \nabla_{n+1}$ ($n \geq 2$) and signs $\{\theta_j\}_{j \in A}$ we have that

$$
\begin{aligned}
\left\| \sum_{j \in A} \theta_j f_j^* \right\|_{Z^*} &\leq \left\| \sum_{j \in A} \theta_j f_j^* \right\|_{Y^*} \\
&= \left\| \sum_{j \in A} \theta_j e_j^* \Big|_{X_A} \right\|_{X_A^*}
\end{aligned}
$$

$$= \left|\left| \left(\sum_{j\in A} \theta_j e_j^* \right) \bigg|_{X_A} \right|\right|_{X_A^*}$$

$$\leq \left|\left| \sum_{i\in A} \theta_i e_i^* \right|\right|_{\ell_2(X^*)}$$

$$= \left|\left| \sum_{i\in A} e_i^* \right|\right|_{\ell_2(X^*)}$$

$$\leq \frac{\sqrt{2}}{c(\log_2 |A|)^\gamma} |A|^{1/2}$$

$$\leq \frac{\sqrt{2}}{c(\frac{n+1}{8} - 2)^\gamma} (2m_{n+1})^{1/2}$$

since $2m_{n+1} \geq |A| \geq m_{n+1} \geq 2^{n+1/8-2}$.

Let $E_n = \left\{ \sum_{j\in A} \theta_j y_j : A \in \nabla_{n+1}, \theta_j = \pm 1 \right\}$. Then,

$$|\beta_n(T) - \beta_{n-1}(T)| \leq 2^{-n-1} \sum_{A\in\nabla_{n+1}} \frac{\sqrt{2}}{c(\frac{n+1}{8} - 2)^\gamma} (2m_{n+1})^{1/2} \sup\{||Tz|| : z \in E_n\}$$

$$\leq 2^{-n-1}(2^{n+1} m_{n+1}^{-1}) \frac{\sqrt{2}}{c(\frac{n+1}{8} - 2)^\gamma} (2m_{n+1})^{1/2} \sup\{||Tz|| : z \in E_n\}$$

$$= \frac{1}{c(\frac{n+1}{8} - 2)^\gamma} 2m_{n+1}^{-1/2} \sup\{||Tz|| : z \in E_n\}.$$

If we put $F_n = \frac{1}{c(\frac{n+1}{8}-2)^\gamma} 2m_{n+1}^{-1/2} E_n$, we see that (iii) holds. So, we are left to show that (iv) holds. Let $A \in \nabla_{n+1}$ $(n \geq 2)$ and $\{\theta_j\}_{j\in A}$ and every $1 \leq k \leq 9$. Define a sequence $\{\alpha_{g_k(j)}\}_{j\in A}$ by $\alpha_{g_k(j)} = \theta_j$ and note that it is well-defined by (iii) of Lemma III.5. Using the fact that $g_k(A) \subset \sigma_m$, where $m = n, n+1$, or $n+2$ (see the statement preceding Lemma III.3), we have

$$\left|\left| \sum_{j\in A} \theta_j f_{g_k(j)} \right|\right| = \left|\left| \sum_{j\in g_k(A)} \alpha_j f_j \right|\right|$$

$$= \left(\sum_{A_0 \in \nabla_m} \left|\left| \sum_{j \in A_0 \cap g_k(A)} \alpha_j e_j \right|\right|^2 \right)^{1/2}$$

$$= \left(\sum_{j \in g_k(A)} ||\alpha_j e_j||^2 \right)^{1/2}$$

$$= \left(\sum_{j \in g_k(A)} |\alpha_j|^2 \right)^{1/2}$$

$$= \left(\sum_{j \in A} |\theta_j|^2 \right)^{1/2}$$

$$= |A|^{1/2}$$

$$\leq (2m_{n+1})^{1/2}$$

and hence

$$\left|\left| \sum_{j \in A} \theta_j y_j \right|\right| = \left|\left| \sum_{k=1}^{9} \lambda_{j,k} \sum_{j \in A} \theta_j f_{g_k(j)} \right|\right|$$

$$\leq \sum_{k=1}^{9} |\lambda_{j,k}| \left|\left| \sum_{j \in A} \theta_j f_{g_k(j)} \right|\right|$$

$$\leq (8+2)(2m_{n+1})^{1/2}$$

$$\leq 15 m_{n+1}^{1/2}.$$

Finally, for every $A \in \nabla_{n+1}$ we get that

$$\sup\{||x|| : x \in F_n\} \leq \frac{1}{c(\frac{n+1}{8} - 2)^\gamma} 2m_{n+1}^{-1/2} 15 m_{n+1}^{1/2}$$

$$= \frac{30}{c(\frac{n+1}{8} - 2)^\gamma}.$$

Since $\sum \frac{30}{c(\frac{n+1}{8}-2)^{\gamma}} < \infty$, we obtain (iv).

CHAPTER V

An Example

Based on Theorem IV.6, the goal of this chapter is to describe a specific infinite dimensional Banach space X for which $\ell_2(X)$ has a subspace without the $C.A.P.$ For the Banach space X, which will be exhibited in Example V.1, we will not be able to verify that it satisfies the statement of Theorem IV.6 since it is difficult to compute the sequence of Euclidean distances $\{d_n(X^*)\}_n$ and, in turn, to show that they are greater than $\{c(\log_2 n)^\beta\}_n$ for some absolute constants $c > 0$ and $\beta > 4$. Instead, we will direct our attention to finding, for each n, a set of normalized 1-unconditional vectors $\{u_1, \ldots, u_n\} \subset \ell_2(X^*)$ satisfying

$$\left\| \sum_{j=1}^{n} u_j \right\| \leq \frac{1}{c(\log_2 n)^\gamma} n^{1/2}$$

for some absolute constants $c > 0$ and $\gamma > 1$, since it will allow us to use the arguments of Theorem IV.6.

We will model our example X after the Banach space constructed by Johnson in [5] , which has the property that all its subspaces have the approximation property; X will be of the form $X = (\sum_{n\geq 1} \bigoplus \ell_{q_n}^{k_n})_{\ell_2}$ with $\{k_n\}_n$ a fast increasing sequence converging to infinity and $\{q_n\}_n$ a sequence converging quickly to 2. While we are not able to show that $\ell_2(X)$ admits a subspace without the $C.A.P.$ for Johnson's space X, our example is not far from it. We will comment more on this point at the end of this chapter.

Example V.1. We will construct our space $X = (\sum_{n\geq 1} \bigoplus \ell_{q_n}^{k_n})_{\ell_2}$ so that we have $X^* = (\sum_{n\geq 1} \bigoplus \ell_{p_n}^{k_n})_{\ell_2}$, where we choose $\{p_n\}_n$ and $\{k_n\}_n$ in the following way: we first pick absolute constants $\beta > 2$ and $\gamma > 1$ and then we proceed by taking $p_1 = 3$ and picking $k_1, p_2, k_2, \ldots, p_n, k_n, \ldots$, in this order such that

$$k_n^{\left|\frac{1}{p_n}-\frac{1}{2}\right|} = 2^{h(n)},$$

where $h(n) \to \infty$ as $n \to \infty$ satisfies $h(n) \geq 1$ and $\left(\frac{\beta h(n)^2}{\left|\frac{1}{p_n}-\frac{1}{2}\right|}\right)^{\gamma} \leq 2^{h(n)}$ (basically p_n determines $h(n)$ which in turn determines k_n). Then choose p_{n+1} such that

$$m(k_n)^{\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|} = 2,$$

where $m(k_n) = k_n^{\beta}$. We notice that $k_n \nearrow \infty$ and $p_n \searrow 2$. Indeed, by our construction

$$k_n^{\frac{1}{h(n)}\left|\frac{1}{p_n}-\frac{1}{2}\right|} = k_n^{\beta\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|} = 2,$$

$$\frac{1}{h(n)}\left|\frac{1}{p_n}-\frac{1}{2}\right| = \beta\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|,$$

$$\frac{\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|}{\left|\frac{1}{p_n}-\frac{1}{2}\right|} = \frac{1}{\beta h(n)}.$$

Since $\frac{1}{\beta h(n)} < \frac{1}{2}$ for all n we have,

$$\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right| < \frac{1}{2}\left|\frac{1}{p_n}-\frac{1}{2}\right| < \cdots < \frac{1}{2^n}\left|\frac{1}{p_1}-\frac{1}{2}\right|.$$

Thus, $p_n \searrow 2$ and as a consequence $k_n \nearrow \infty$.

Given s, let $\alpha(s)$ be the smallest constant α for which we can find 1-unconditional normalized vectors $\{u_1, \ldots, u_s\} \subset \ell_2(X^*)$ satisfying

$$\left|\left|\sum_{j=1}^{s} u_j\right|\right| \leq \alpha \cdot s^{1/2}.$$

In order to use the arguments of Theorem IV.6 and conclude that $\ell_2(X)$ has a subspace without the $C.A.P.$, we are left to show that for each $s \geq 1$, $\alpha(s) \leq \frac{1}{c(\log_2 s)^\gamma}$ for some absolute constant $c > 0$.

Let s be arbitrary and find n such that $k_n < s \leq k_{n+1}$ where for convenience we denote $k_0 = 0$. For $j \in \{1, 2, \ldots, s\}$, let $y_j = e_j$ where $\{e_j\}_{j=1}^{k_{n+1}} \subset \ell_{p_{n+1}}^{k_{n+1}}$ forms the unit vector basis. Clearly, $\{y_1, y_2, \ldots, y_s\}$ are 1-unconditional normalized vectors. Also,

$$\begin{aligned}\left|\left|\sum_{y=1}^{s} y_j\right|\right| &= s^{\frac{1}{p_{n+1}}} \\ &= s^{\frac{1}{p_{n+1}}-\frac{1}{2}} \cdot s^{\frac{1}{2}} \\ &= s^{-\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|} \cdot s^{\frac{1}{2}}\end{aligned}$$

Thus, $\alpha(s) \leq s^{-\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|}$.

Without loss of generality, suppose next that s is divisible by k_n. Then, there is a positive integer l such that $s = lk_n$. We will define $\{x_i\}_{i=1}^{s} \subset \ell_2(\ell_{p_n}^{k_n}) \subset \ell_2(X^*)$ in the following way. First, for $i \in \{1, 2, \ldots, k_n\}$ we define $x_i = (e_i, 0, 0, \ldots)$, where $\{e_i\}_{i=1}^{k_n}$ is the unit vector basis in $\ell_{p_n}^{k_n}$. Next, for $i \in \{k_n + 1, \ldots, 2k_n\}$ we define $x_i = (0, e_{i-k_n}, 0, \ldots)$. Then, continuing in this way we can define x_i for $i \in \{(m-1)k_n + 1, \ldots, mk_n\}$ with $m = 3, 4, \ldots, l$. Namely, we define x_i by $x_i = (0, 0, \ldots, e_{i-(m-1)k_n}, 0, \ldots)$, where $e_{i-(m-1)k_n}$ is in the mth entry. Then, $\{x_i\}_{i=1}^{s} \subset \ell_2(\ell_{p_n}^{k_n})$ are 1-unconditional normalized vectors and

$$\left|\left|\sum_{j=1}^{s} x_j\right|\right|_{\ell_2(\ell_{p_n}^{k_n})} = \left|\left|(x_1 + \cdots + x_{k_n}) + (x_{k_n+1} + \cdots + x_{2k_n}) + \cdots + (x_{(l-1)k_n+1} + \cdots + x_{lk_n})\right|\right|$$

$$= \left(k_n^{\frac{1}{p_n}\cdot 2} + k_n^{\frac{1}{p_n}\cdot 2} + \cdots + k_n^{\frac{1}{p_n}\cdot 2} \right)^{1/2}$$

$$= (k_n^{\frac{2}{p_n}} \cdot l)^{1/2}$$

$$= k_n^{\frac{1}{p_n}} \cdot l^{1/2}$$

$$= k_n^{\frac{1}{p_n}-\frac{1}{2}}(k^{1/2} \cdot l^{1/2})$$

$$= k_n^{-|\frac{1}{p_n}-\frac{1}{2}|} \cdot s^{1/2}$$

Thus, $\alpha(s) \leq k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}$.

We now have that $\alpha(s) \leq \min\{s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}, k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}\}$, whenever $k_n < s \leq k_{n+1}$, and we will finish the example by showing that the $\min\{s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}, k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}\} \leq \frac{1}{(\log_2 s)^\gamma}$.

Let $k_n < s \leq k_{n+1}$. We first notice that the intersection of the portion of the graph $s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}$ and the constant function $k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}$ is when $s = m(k_n)^{h(n)} = k_n^{\beta h(n)}$, which we will denote as $\widetilde{k}_n$. Indeed, $k_n^{-|\frac{1}{p_n}-\frac{1}{2}|} = 2^{-h(n)} = \big(m(k_n)^{h(n)}\big)^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}$. Now, for $k_n < s \leq k_{n+1}$ we can graph $\min\{s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}, k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}\}$ as a function of s, where its graph will be the constant function $k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}$ on the interval $(k_n, \widetilde{k}_n]$ and $s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}$ on the interval $(\widetilde{k}_n, k_{n+1}]$. By our construction, we have that $\frac{h(n)}{|\frac{1}{p_n}-\frac{1}{2}|} = \log_2 k_n$ and $2^{h(n)} = k_n^{\beta h(n)|\frac{1}{p_{n+1}}-\frac{1}{2}|}$. Since $\left(\frac{\beta h(n)^2}{|\frac{1}{p_n}-\frac{1}{2}|}\right)^\gamma \leq 2^{h(n)}$ we get that

$$(\log_2 k_n^{\beta h(n)})^\gamma \leq k_n^{\beta h(n)|\frac{1}{p_{n+1}}-\frac{1}{2}|},$$

which implies

$$\alpha(\widetilde{k}_n) = \frac{1}{\widetilde{k}_n^{|\frac{1}{p_{n+1}}-\frac{1}{2}|}} \leq \frac{1}{(\log_2 \widetilde{k}_n)^\gamma}.$$

Hence, for $s \in (k_n, \widetilde{k}_n]$, we have $k_n^{-|\frac{1}{p_n}-\frac{1}{2}|} \leq \frac{1}{(\log_2 s)^\gamma}$ since the latter function is decreasing and the former(constant) function is equal to $\widetilde{k}_n^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}$. Let $s \in (\widetilde{k}_n, k_{n+1}]$ and write $s = \widetilde{k}_n^j$ for some $j > 1$. Then, using the fact that $j(\log_2 \widetilde{k}_n) < (\log_2 \widetilde{k}_n)^j$ we have that

$$\begin{aligned}\frac{1}{s^{|\frac{1}{p_{n+1}}-\frac{1}{2}|}} &= \frac{1}{\widetilde{k}_n^{j|\frac{1}{p_{n+1}}-\frac{1}{2}|}} \\ &= \left(\frac{1}{\widetilde{k}_n^{|\frac{1}{p_{n+1}}-\frac{1}{2}|}}\right)^j \\ &< \left(\frac{1}{\log_2 \widetilde{k}_n}\right)^{j\gamma} \\ &< \left(\frac{1}{j\log_2 \widetilde{k}_n}\right)^{\gamma} \\ &< \frac{1}{(\log_2 s)^\gamma}.\end{aligned}$$

Therefore we obtain $\alpha(s) \leq \min\{s^{-|\frac{1}{p_{n+1}}-\frac{1}{2}|}, k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}\} \leq \frac{1}{(\log_2 s)^\gamma}$. □

Remark. If s is not divisible by k_n in the above example, then we will have

$$\alpha(s) \leq \min\{\sqrt{2}s^{-\left|\frac{1}{p_{n+1}}-\frac{1}{2}\right|}, \sqrt{2}k_n^{-|\frac{1}{p_n}-\frac{1}{2}|}\} \leq \frac{1}{c(\log_2 s)^\gamma},$$

but with an absolute constant $c = \frac{1}{\sqrt{2}}$.

Remark. Johnson's space was constructed with $m(k_n) = 5^{5^{k_n}}$. In this case, by an argument similar to the previous discussion, we get

$$\alpha(s) \leq \frac{1}{(\log_5(\log_5(\log_5 s)))^\gamma}.$$

The only difference is that now $\widetilde{k}_n = m(k_n)^{h(n)}$ is larger than before, namely $\widetilde{k}_n = (5^{5^{k_n}})^{h(n)}$, which forces

$$\alpha(\widetilde{k}_n) \leq \frac{1}{(\log_5(\log_5(\log_5 \widetilde{k}_n)))^\gamma}$$

for a suitable choice of $h(n)$. Subsequently,

$$\alpha(s) \leq \frac{1}{(\log_5(\log_5(\log_5 s)))^\gamma}.$$

Therefore, in order to obtain that $\ell_2(X)$ has a subspace without the $C.A.P.$ for Johnson's space X, we would need to prove a similar statement as in Theorem IV.6 but involving more iterates of log.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] R. Anisca, *Unconditional decompositions in subspaces of $\ell_2(X)$*, Positivity **8** (2004), 423–441.

[2] B. Bollobas, *Linear Analysis An Introductory Course*, Cambridge University Press, Cambridge, UK, 1999.

[3] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. **130** (1973), 309–317.

[4] M. Fabian et al., *Functional Analysis and Infinite Dimensional Geometry*, Springer-Verlag, New York, 2001.

[5] W.B. Johnson, *Banach spaces all of whose subspaces have the approximation property*, Special Topics in Applied Mathematics (North Holland), 1980, pp. 15–25.

[6] R. Komorowski and N. Tomczak-Jaegermann, *Subspaces of $\ell_2(X)$ and Rad(X) without unconditional structure*, Studia Matematica **149** (2002), 1–21.

[7] V. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Springer-Verlag, New York, 1986.

[8] N.J. Nielsen and N. Tomczak-Jaegermann, *Banach lattices with property (H) and weak Hilbert spaces*, Illinois J. of Math. **36** (1992), 345–371.

[9] A. Szankowski, *Subspaces without the approximation property*, Israel J. Math. **30** (1978), 123–127.

[10] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge University Press, Cambridge, UK, 1991.